P.1 Real Numbers

Real Numbers ▫ Ordering Real Numbers ▫ Absolute Value and Distance ▫ Algebraic Expressions ▫ Basic Rules of Algebra

See Exercises 67–70 on page 11 for an example of how real numbers and absolute value are used to solve a budget variance problem.

Real Numbers

Real numbers are used in everyday life to describe quantities such as age, miles per gallon, container size, and population. To represent real numbers, you can use symbols such as

$$9,\ 0,\ \frac{4}{3},\ 0.666\ .\ .\ .\ ,\ 28.21,\ \sqrt{2},\ \pi,\ \text{and}\ \sqrt[3]{-32}.$$

Here are some important subsets of the real numbers.

$\{1, 2, 3, 4, \ldots\}$	Set of natural numbers
$\{0, 1, 2, 3, 4, \ldots\}$	Set of whole numbers
$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$	Set of integers

A real number is **rational** if it can be written as the ratio p/q of two integers, where $q \neq 0$. For instance, the numbers

$$\frac{1}{3} = 0.3333\ .\ .\ .\ ,\quad \frac{1}{8} = 0.125,\quad \text{and}\quad \frac{125}{111} = 1.126126\ .\ .\ .$$

are rational. The decimal representation of a rational number either *repeats* (as in 3.1454545 . . .) or *terminates* (as in $\frac{1}{2} = 0.5$). A real number that cannot be written as the ratio of two integers is called **irrational.** Irrational numbers have infinite *nonrepeating* decimal representations. For instance, the numbers

$$\sqrt{2} \approx 1.4142136 \quad \text{and} \quad \pi \approx 3.1415927$$

are irrational. (The symbol $\approx$ means "is approximately equal to.")

Real numbers are represented graphically by a **real number line.** The point 0 on the real number line is the **origin.** Numbers to the right of 0 are positive, and numbers to the left of 0 are negative, as shown in Figure P.1. The term **nonnegative** describes a number that is either positive or zero.

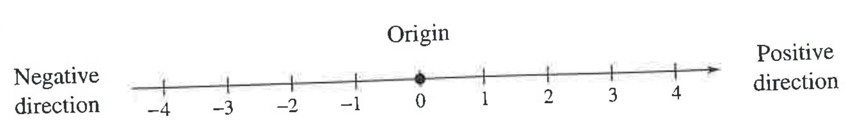
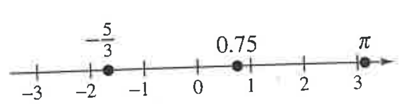
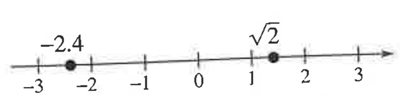


FIGURE P.1 The Real Number Line

As illustrated in Figure P.2, there is a *one-to-one correspondence* between real numbers and points on the real number line.



Every real number corresponds to exactly one point on the real number line.



Every point on the real number line corresponds to exactly one real number.

FIGURE P.2 One-to-One Correspondence

Ordering Real Numbers

One important property of real numbers is that they are **ordered.**

DEFINITION OF ORDER ON THE REAL NUMBER LINE

If a and b are real numbers, a is **less than** b if $b - a$ is positive. This order is denoted by the **inequality**

$$a < b.$$

This can also be described by saying that b is **greater than** a and writing $b > a$. The inequality $a \le b$ means that a is **less than or equal to** b, and the inequality $b \ge a$ means that b is **greater than or equal to** a. The symbols $<$, $>$, $\le$, and $\ge$ are **inequality symbols.**

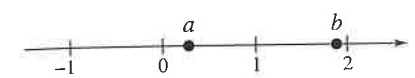


FIGURE P.3 $a < b$ if and only if a lies to the left of b.

Geometrically, this definition implies that $a < b$ if and only if a lies to the *left* of b on the real number line, as shown in Figure P.3.

EXAMPLE 1 Interpreting Inequalities

a. The inequality $x \le 2$ denotes all real numbers less than or equal to 2, as shown in Figure P.4(a).

b. The inequality $-2 \le x < 3$ means that $x \ge -2$ *and* $x < 3$. The "double inequality" denotes all real numbers between -2 and 3, including -2 but *not* including 3, as shown in Figure P.4(b).

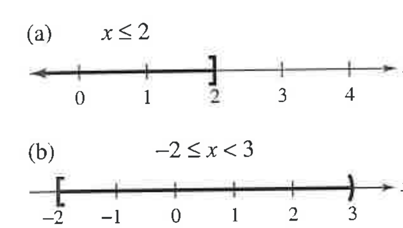
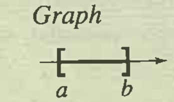
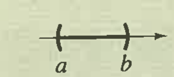
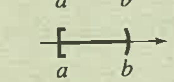
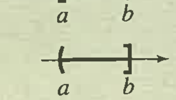


FIGURE P.4

Inequalities can be used to describe subsets of real numbers called **intervals.**

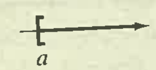
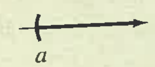
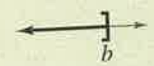
NOTE In the bounded intervals at the right, the real numbers a and b are the **endpoints** of each interval.

BOUNDED INTERVALS ON THE REAL NUMBER LINE

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, b]$	Closed	$a \le x \le b$	
(a, b)	Open	$a < x < b$	
$[a, b)$	Half-open	$a \le x < b$	
$(a, b]$	Half-open	$a < x \le b$	

NOTE The symbols ∞, **positive infinity,** and $-\infty$, **negative infinity,** do not represent real numbers. They are simply convenient symbols used to describe the unboundedness of an interval such as $(1, \infty)$ or $(-\infty, 3]$.

UNBOUNDED INTERVALS ON THE REAL NUMBER LINE

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, \infty)$	Half-open	$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$	Half-open	$x \leq b$	
$(-\infty, b)$	Open	$x < b$	
$(-\infty, \infty)$	Entire real line		

EXAMPLE 2 *Using Inequalities to Represent Intervals*

Use inequality notation to describe each of the following.

a. c is at most 2.

b. All x in the interval $(-3, 5]$

Solution

a. The statement "c is at most 2" can be represented by $c \leq 2$.

b. "All x in the interval $(-3, 5]$" can be represented by $-3 < x \leq 5$.

EXAMPLE 3 *Interpreting Intervals*

Give a verbal description of each interval.

a. $(-1, 0)$ **b.** $[2, \infty)$ **c.** $(-\infty, 0)$

Solution

a. This interval consists of all real numbers that are greater than -1 and less than 0.

b. This interval consists of all real numbers that are greater than or equal to 2.

c. This interval consists of all negative real numbers.

The **Law of Trichotomy** states that for any two real numbers a and b, *precisely* one of three relationships is possible:

$$a = b, \quad a < b, \quad \text{or} \quad a > b. \qquad \text{Law of Trichotomy}$$

Exploration

Absolute value expressions can be evaluated on a graphing utility. To evaluate $|-4|$ with a *TI-83*, use these keystrokes:

[MATH] (NUM) (1:abs ( )

[ ] 4 [ENTER].

To evaluate $|-4|$ with a *TI-82*, use these keystrokes:

[ABS] [ ] 4 [ENTER].

When evaluating an expression such as $|3 - 8|$, parentheses should surround the entire expression. Evaluate each expression below. What can you conclude?

a. $|6|$ **b.** $|-1|$

c. $|5 - 2|$ **d.** $|2 - 5|$

NOTE The absolute value of a real number is either positive or zero. Moreover, 0 is the only real number whose absolute value is 0. Thus, $|0| = 0$.

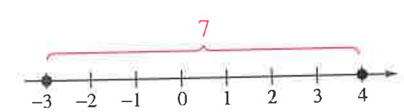


FIGURE P.5 The distance between -3 and 4 is 7.

Absolute Value and Distance

The **absolute value** of a real number is its *magnitude.*

DEFINITION OF ABSOLUTE VALUE

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

Notice from this definition that the absolute value of a real number is never negative. For instance if $a = -5$, then $|-5| = -(-5) = 5$.

EXAMPLE 4 *Evaluating the Absolute Value of a Number*

Evaluate $\dfrac{|x|}{x}$ for (a) $x > 0$ and (b) $x < 0$.

Solution

a. If $x > 0$, then $|x| = x$ and $\dfrac{|x|}{x} = \dfrac{x}{x} = 1$.

b. If $x < 0$, then $|x| = -x$ and $\dfrac{|x|}{x} = \dfrac{-x}{x} = -1$.

PROPERTIES OF ABSOLUTE VALUES

1. $|a| \geq 0$ **2.** $|-a| = |a|$

3. $|ab| = |a||b|$ **4.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

Absolute value can be used to define the distance between two numbers on the real number line. For instance, the distance between -3 and 4 is $|-3 - 4| = |-7| = 7$, as shown in Figure P.5.

DISTANCE BETWEEN TWO POINTS ON THE REAL LINE

Let a and b be real numbers. The **distance between a and b** is

$$d(a, b) = |b - a| = |a - b|.$$

Algebraic Expressions

One characteristic of algebra is the use of letters to represent numbers. The letters are **variables,** and combinations of letters and numbers are **algebraic expressions.** Here are a few examples of algebraic expressions.

$$5x, \quad 2x - 3, \quad \frac{4}{x^2 + 2}, \quad 7x + y$$

Definition of an Algebraic Expression

A collection of letters **(variables)** and real numbers **(constants)** combined using the operations of addition, subtraction, multiplication, division, and exponentiation is an **algebraic expression.**

The **terms** of an algebraic expression are those parts that are separated by *addition.* For example,

$$x^2 - 5x + 8 = x^2 + (-5x) + 8$$

has three terms: x^2 and $-5x$ are the **variable terms** and 8 is the **constant term.** The numerical factor of a variable term is the **coefficient** of the variable term. For instance, the coefficient of $-5x$ is -5, and the coefficient of x^2 is 1.

To **evaluate** an algebraic expression, substitute numerical values for each of the variables in the expression. Here are two examples.

Expression	*Value of Variable*	*Substitute*	*Value of Expression*
$-3x + 5$	$x = 3$	$-3(3) + 5$	$-9 + 5 = -4$
$3x^2 + 2x - 1$	$x = -1$	$3(-1)^2 + 2(-1) - 1$	$3 - 2 - 1 = 0$

Study Tip

When evaluating an algebraic expression, the Substitution Principle is used. It states, "If $a = b$, then a can be replaced by b in any expression involving a." In the first evaluation shown at the right, for instance, 3 is *substituted* for x in the expression $-3x + 5$.

The French mathematician Nicolas Chuquet (ca. 1500) wrote Triparty en la science des nombres, *in which a form of exponent notation was used. Our expressions* $6x^3$ *and* $10x^2$ *were written as* $.6.^3$ *and* $.10.^2$*. Zero and negative exponents were also represented, so* x^0 *would be written as* $.1.^0$ *and* $3x^{-2}$ *as* $.3.^{2.m}$*. Chuquet wrote that* $.72.^1$ *divided by* $.8.^3$ *is* $.9.^{2.m}$*. That is,* $72x \div 8x^3 = 9x^{-2}$.

Basic Rules of Algebra

There are four arithmetic operations with real numbers: **addition, multiplication, subtraction,** and **division,** denoted by the symbols $+$, $\times$ or $\cdot$, $-$, and $\div$. Of these, addition and multiplication are the two primary operations. Subtraction and division are the inverse operations of addition and multiplication, respectively.

Subtraction

$$a - b = a + (-b)$$

Division

$$\text{If } b \neq 0, \text{ then } a \div b = a\left(\frac{1}{b}\right) = \frac{a}{b}.$$

In these definitions, $-b$ is the **additive inverse** (or opposite) of b, and $1/b$ is the **multiplicative inverse** (or reciprocal) of b. In the fractional form a/b, a is the **numerator** of the fraction and b is the **denominator.**

Be sure you see that the following **basic rules of algebra** are true for variables and algebraic expressions as well as for real numbers. Try to formulate a verbal description of each property. For instance, the first property states that *the order in which two real numbers are added does not affect their sum.*

Basic Rules of Algebra

Let a, b, and c be real numbers, variables, or algebraic expressions.

Property		*Example*
Commutative Property of Addition:	$a + b = b + a$	$4x + x^2 = x^2 + 4x$
Commutative Property of Multiplication:	$ab = ba$	$(4 - x)x^2 = x^2(4 - x)$
Associative Property of Addition:	$(a + b) + c = a + (b + c)$	$(x + 5) + x^2 = x + (5 + x^2)$
Associative Property of Multiplication:	$(ab)c = a(bc)$	$(2x \cdot 3y)(8) = (2x)(3y \cdot 8)$
Distributive Properties:	$a(b + c) = ab + ac$	$3x(5 + 2x) = 3x \cdot 5 + 3x \cdot 2x$
	$(a + b)c = ac + bc$	$(y + 8)y = y \cdot y + 8 \cdot y$
Additive Identity Property:	$a + 0 = a$	$5y^2 + 0 = 5y^2$
Multiplicative Identity Property:	$a \cdot 1 = a$	$(4x^2)(1) = 4x^2$
Additive Inverse Property:	$a + (-a) = 0$	$5x^3 + (-5x^3) = 0$
Multiplicative Inverse Property:	$a \cdot \frac{1}{a} = 1, \quad a \neq 0$	$(x^2 + 4)\left(\frac{1}{x^2 + 4}\right) = 1$

NOTE Because subtraction is defined as "adding the opposite," the Distributive Properties are also true for subtraction. For instance, the "subtraction form" of $a(b + c) = ab + ac$ is

$$a(b - c) = ab - ac.$$

NOTE Be sure you see the difference between the *opposite of a number* and a *negative number.* If a is already negative, then its opposite, $-a$, is positive. For instance, if $a = -5$, then $-a = -(-5) = 5$.

As well as formulating a verbal description for each of the following basic properties of negation, zero, and fractions, try to gain an *intuitive sense* for the validity of each.

Properties of Negation

Let a and b be real numbers, variables, or algebraic expressions.

Property	*Example*
1. $(-1)a = -a$	$(-1)7 = -7$
2. $-(-a) = a$	$-(-6) = 6$
3. $(-a)b = -(ab) = a(-b)$	$(-5)3 = -(5 \cdot 3) = 5(-3)$
4. $(-a)(-b) = ab$	$(-2)(-x) = 2x$
5. $-(a + b) = (-a) + (-b)$	$-(x + 8) = (-x) + (-8)$ $= -x - 8$

NOTE The "or" in the Zero-Factor Property includes the possibility that either or both factors may be zero. This is an **inclusive or,** and it is the way the word "or" is generally used in mathematics.

PROPERTIES OF ZERO

Let a and b be real numbers, variables, or algebraic expressions.

1. $a + 0 = a$ and $a - 0 = a$
2. $a \cdot 0 = 0$
3. $\dfrac{0}{a} = 0, \quad a \neq 0$
4. $\dfrac{a}{0}$ is undefined.
5. **Zero-Factor Property:** If $ab = 0$, then $a = 0$ or $b = 0$.

NOTE In Property 1, the phrase "if and only if" implies two statements. One statement is: If $a/b = c/d$, then $ad = bc$. The other statement is: If $ad = bc$, where $b \neq 0$ and $d \neq 0$, then $a/b = c/d$.

PROPERTIES OF FRACTIONS

Let a, b, c, and d be real numbers, variables, or algebraic expressions such that $b \neq 0$ and $d \neq 0$.

1. **Equivalent Fractions:** $\dfrac{a}{b} = \dfrac{c}{d}$ if and only if $ad = bc$.
2. **Rules of Signs:** $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$ and $\dfrac{-a}{-b} = \dfrac{a}{b}$
3. **Generate Equivalent Fractions:** $\dfrac{a}{b} = \dfrac{ac}{bc}, \quad c \neq 0$
4. **Add or Subtract with Like Denominators:** $\dfrac{a}{b} \pm \dfrac{c}{b} = \dfrac{a \pm c}{b}$
5. **Add or Subtract with Unlike Denominators:** $\dfrac{a}{b} \pm \dfrac{c}{d} = \dfrac{ad \pm bc}{bd}$
6. **Multiply Fractions:** $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$
7. **Divide Fractions:** $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}, \quad c \neq 0$

EXAMPLE 5 *Properties of Fractions*

a. $\dfrac{x}{5} = \dfrac{3 \cdot x}{3 \cdot 5} = \dfrac{3x}{15}$ — Generate equivalent fractions.

b. $\dfrac{x}{3} + \dfrac{2x}{5} = \dfrac{5 \cdot x + 3 \cdot 2x}{15}$ — Add fractions with unlike denominators.

c. $\dfrac{7}{x} \div \dfrac{3}{2} = \dfrac{7}{x} \cdot \dfrac{2}{3} = \dfrac{14}{3x}$ — Divide fractions.

PROPERTIES OF EQUALITY

Let a, b, and c be real numbers, variables, or algebraic expressions.

1. If $a = b$, then $a + c = b + c$. — Add c to both sides.
2. If $a = b$, then $ac = bc$. — Multiply both sides by c.
3. If $a + c = b + c$, then $a = b$. — Subtract c from both sides.
4. If $ac = bc$ and $c \neq 0$, then $a = b$. — Divide both sides by c.

If a, b, and c are integers such that $ab = c$, then a and b are **factors** or **divisors** of c. A **prime number** is an integer that has exactly two positive factors: itself and 1. For example, 2, 3, 5, 7, and 11 are prime numbers. The numbers 4, 6, 8, 9, and 10 are **composite** because they can be written as the product of two or more prime numbers. The number 1 is neither prime nor composite. The **Fundamental Theorem of Arithmetic** states that every positive integer greater than 1 can be written as the product of prime numbers in precisely one way (disregarding order). For instance, the *prime factorization* of 24 is $24 = 2 \cdot 2 \cdot 2 \cdot 3$.

When adding or subtracting fractions with unlike denominators, you have two options. You can use Property 5 of fractions as in Example 5(b), or you can rewrite the fractions with like denominators. Here is an example.

$$\frac{2}{15} - \frac{5}{9} + \frac{4}{5} = \frac{2(3)}{15(3)} - \frac{5(5)}{9(5)} + \frac{4(9)}{5(9)} \qquad \text{The LCD is 45.}$$
$$= \frac{6 - 25 + 36}{45}$$
$$= \frac{17}{45}$$

GROUP ACTIVITY

DECIMAL APPROXIMATIONS OF IRRATIONAL NUMBERS

At the beginning of this section, it was pointed out that $\sqrt{2}$ is not a rational number. There are, however, rational numbers whose squares are very close to 2. For instance, if you square the rational number

$$\frac{140}{99}$$

you obtain 1.9998. Try finding other rational numbers whose squares are even closer to 2. Write a short paragraph explaining how you obtained the numbers.

P.1 Exercises

In Exercises 1–6, determine which numbers are (a) natural numbers, (b) integers, (c) rational numbers, and (d) irrational numbers.

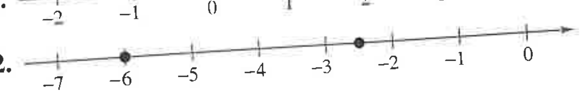
1. $-9, -\frac{7}{2}, 5, \frac{2}{3}, \sqrt{2}, 0, 1$
2. $\sqrt{5}, -7, -\frac{7}{3}, 0, 3.12, \frac{5}{4}$
3. $2.01, 0.666\ldots, -13, 0.010110111\ldots$
4. $2.30300030003\ldots, 0.7575, -4.63, \sqrt{10}$
5. $-\pi, -\frac{1}{3}, \frac{6}{3}, \frac{1}{2}\sqrt{2}, -7.5$
6. $25, -17, -\frac{12}{5}, \sqrt{9}, 3.12, \frac{1}{2}\pi$

In Exercises 7–10, use a calculator to find the decimal form of the rational number. If it is a nonterminating decimal, write the repeating pattern.

7. $\frac{5}{8}$
8. $\frac{1}{3}$
9. $\frac{41}{333}$
10. $\frac{6}{11}$

In Exercises 11 and 12, approximate the numbers and place the correct symbol (< or >) between them.

11. 

12. 

In Exercises 13–18, plot the two real numbers on the real number line. Then place the appropriate inequality sign (< or >) between them.

13. $\frac{3}{2}, 7$
14. $-3.5, 1$
15. $-4, -8$
16. $1, \frac{16}{3}$
17. $\frac{5}{6}, \frac{2}{3}$
18. $-\frac{8}{7}, -\frac{3}{7}$

In Exercises 19–28, verbally describe the subset of real numbers represented by the inequality. Then sketch the subset on the real number line. State whether the interval is bounded or unbounded.

19. $x \le 5$
20. $x \ge -2$
21. $x < 0$
22. $x > 3$
23. $x \ge 4$
24. $x < 2$
25. $-2 < x < 2$
26. $0 \le x \le 5$
27. $-1 \le x < 0$
28. $0 < x \le 6$

In Exercises 29 and 30, use a calculator to order the numbers from smallest to largest.

29. $\frac{7071}{5000}, \frac{584}{413}, \sqrt{2}, \frac{47}{33}, \frac{127}{90}$
30. $\frac{26}{15}, \sqrt{3}, 1.7320, \frac{381}{220}, \sqrt{10} - \sqrt{2}$

In Exercises 31–36, use inequality notation to describe the set.

31. x is negative.
32. z is at least 10.
33. y is nonnegative.
34. y is no more than 25.
35. The person's age A is at least 30.
36. The annual rate of inflation r is expected to be at least 2.5%, but no more than 5%.

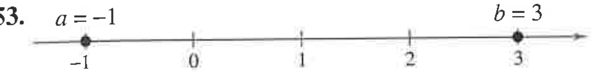
In Exercises 37–46, evaluate the expression.

37. $|-10|$
38. $|0|$
39. $|3 - \pi|$
40. $|4 - \pi|$
41. $\dfrac{-5}{|-5|}$
42. $-3 - |-3|$
43. $-3|-3|$
44. $|-1| - |-2|$
45. $-|16.25| + 20$
46. $2|33|$

In Exercises 47–52, place the correct symbol (<, >, or =) between the pair of real numbers.

47. $|-3| \ \square \ -|-3|$
48. $|-4| \ \square \ |4|$
49. $-5 \ \square \ -|5|$
50. $-|-6| \ \square \ |-6|$
51. $-|-2| \ \square \ -|2|$
52. $-(-2) \ \square \ -2$

In Exercises 53–60, find the distance between a and b.

53. 

54. 

55. 

56. 

57. $a = 126, b = 75$
58. $a = -126, b = -75$
59. $a = \frac{16}{5}, b = \frac{112}{75}$
60. $a = 9.34, b = -5.65$

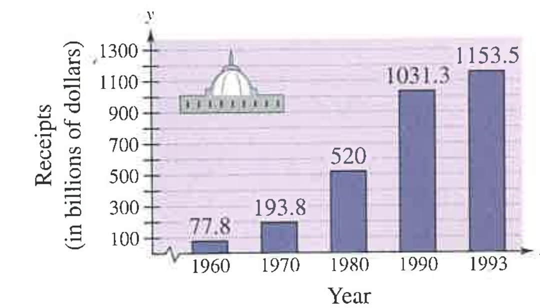
In Exercises 61–66, use absolute value notation to describe the situation.

61. The distance between x and 5 is no more than 3.
62. The distance between x and -10 is at least 6.
63. While traveling, you pass milepost 7, then milepost 18. How far do you travel during that time period?
64. While traveling, you pass milepost 103, then milepost 86. How far do you travel during that time period?
65. y is at least six units from 0.
66. y is at most two units from a.

Budget Variance **In Exercises 67–70, the accounting department of a company is checking to see whether the actual expenses of a department differ from the budgeted expenses by more than \$500 or by more than 5%. Fill in the missing parts of the table, and determine whether the actual expense passes the "budget variance test."**

	Budgeted Expense, b	Actual Expense, a	$\|a - b\|$	$0.05b$
67. Wages	\$112,700	\$113,356		
68. Utilities	\$9400	\$9772		
69. Taxes	\$37,640	\$37,335		
70. Insurance	\$2575	\$2613		

Federal Deficit **In Exercises 71–74, use the bar graph, which shows the receipts of the federal government (in billions of dollars) for selected years from 1960 through 1993. In each exercise you are given the outlay of the federal government. Find the magnitude of the surplus or deficit for the year.** (Source: U.S. Treasury Department)



	Income, y	Outlay, x	$\|y - x\|$
71. 1960		\$92.2 billion	
72. 1980		\$590.9 billion	
73. 1990		\$1252.7 billion	
74. 1993		\$1408.2 billion	

75. ***Exploration*** Consider $|u + v|$ and $|u| + |v|$.

(a) Are the values of the expressions always equal? If not, under what conditions are they unequal?

(b) If the two expressions are not equal for certain values of u and v, is one of the expressions always greater than the other? Explain.

76. ***Think About It*** Is there a difference between saying that a real number is positive and saying that a real number is nonnegative? Explain.

In Exercises 77–80, identify the terms of the expression.

77. $7x + 4$

78. $3x^2 - 8x - 11$

79. $4x^3 + x - 5$

80. $3x^4 + 3x^3$

In Exercises 81–86, evaluate the expression for the values of x. (If not possible, state the reason.)

	Expression	*Values*	
81.	$4x - 6$	(a) $x = -1$	(b) $x = 0$
82.	$9 - 7x$	(a) $x = -3$	(b) $x = 3$
83.	$x^2 - 3x + 4$	(a) $x = -2$	(b) $x = 2$
84.	$-x^2 + 5x - 4$	(a) $x = -1$	(b) $x = 1$
85.	$\dfrac{x+1}{x-1}$	(a) $x = 1$	(b) $x = -1$
86.	$\dfrac{x}{x+2}$	(a) $x = 2$	(b) $x = -2$

In Exercises 87–96, identify the rule(s) of algebra illustrated by the equation.

87. $x + 9 = 9 + x$

88. $2\left(\frac{1}{2}\right) = 1$

89. $\dfrac{1}{h+6}(h + 6) = 1, \quad h \neq -6$

90. $(x + 3) - (x + 3) = 0$

91. $2(x + 3) = 2x + 6$

92. $(z - 2) + 0 = z - 2$

93. $1 \cdot (1 + x) = 1 + x$

94. $x + (y + 10) = (x + y) + 10$

95. $x(3y) = (x \cdot 3)y = (3x)y$

96. $\frac{1}{7}(7 \cdot 12) = \left(\frac{1}{7} \cdot 7\right)12 = 1 \cdot 12 = 12$

In Exercises 97–100, evaluate the expression. (If not possible, state the reason.)

97. $\dfrac{81 - (90 - 9)}{5}$

98. $10(23 - 30 + 7)$

99. $\dfrac{8 - 8}{-9 + (6 + 3)}$

100. $15 - \dfrac{3 - 3}{5}$

In Exercises 101–110, perform the operations. (Write fractional answers in reduced form.)

101. $(4 - 7)(-2)$

102. $\dfrac{27 - 35}{4}$

103. $\frac{3}{16} + \frac{5}{16}$

104. $\frac{6}{7} - \frac{4}{7}$

105. $\frac{5}{8} - \frac{5}{12} + \frac{1}{6}$

106. $\frac{10}{11} + \frac{6}{33} - \frac{13}{66}$

107. $\frac{4}{5} \cdot \frac{1}{2} \cdot \frac{3}{4}$

108. $\frac{11}{16} \div \frac{3}{4}$

109. $12 \div \frac{1}{4}$

110. $\left(\frac{3}{5} \div 3\right) - \left(6 \cdot \frac{4}{8}\right)$

In Exercises 111–114, use a calculator to evaluate the expression. (Round your answer to two decimal places.)

111. $-3 + \frac{3}{7}$

112. $3\left(-\frac{5}{12} + \frac{3}{8}\right)$

113. $\dfrac{11.46 - 5.37}{3.91}$

114. $\dfrac{\frac{1}{5}(-8 - 9)}{-\frac{1}{3}}$

115. Use a calculator to complete the table.

n	1	0.5	0.01	0.0001	0.000001
$5/n$					

116. ***Think About It*** Use the result of Exercise 115 to make a conjecture about the value of $5/n$ as n approaches 0.

117. Use a calculator to complete the table.

n	1	10	100	10,000	100,000
$5/n$					

118. ***Think About It*** Use the result of Exercise 117 to make a conjecture about the value of $5/n$ as n increases without bound.

P.2 Exponents and Radicals

See Exercise 107 on page 26 for an example of how exponents can be used to find the annual depreciation rate in a declining balances problem.

Exponents and Properties ▫ *Scientific Notation* ▫ *Radicals and Their Properties* ▫ *Simplifying Radicals* ▫ *Rationalizing Denominators and Numerators* ▫ *Rational Exponents* ▫ *Radicals and Calculators*

Exponents

Repeated *multiplications* can be written in **exponential form.**

Repeated Multiplication	*Exponential Form*
$a \cdot a \cdot a \cdot a \cdot a$	a^5
$(-4)(-4)(-4)$	$(-4)^3$
$(2x)(2x)(2x)(2x)$	$(2x)^4$

In general, if a is a real number and n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}$$

where n is the **exponent** and a is the **base.** The expression a^n is read "a to the nth **power.**"

Exploration

Use the definition of exponential form to write each of the expressions as a single power of 2. From the results, can you find a general rule for simplifying expressions of the forms

$$a^m a^n \text{ and } \frac{a^m}{a^n}?$$

a. $2^3 \cdot 2^2$

b. $2^5 \cdot 2^1$

c. $\dfrac{2^3}{2^2}$

d. $\dfrac{2^3}{2^1}$

PROPERTIES OF EXPONENTS

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (All denominators and bases are nonzero.)

Property	*Example*
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6 = 729$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^7}{x^4} = x^{7-4} = x^3$
3. $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$	$y^{-4} = \dfrac{1}{y^4} = \left(\dfrac{1}{y}\right)^4$
4. $a^0 = 1, \quad a \neq 0$	$(x^2 + 1)^0 = 1$
5. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3x^3 = 125x^3$
6. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12} = \dfrac{1}{y^{12}}$
7. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{2}{x}\right)^3 = \dfrac{2^3}{x^3} = \dfrac{8}{x^3}$
8. $\lvert a^2 \rvert = \lvert a \rvert^2 = a^2$	$\lvert(-2)^2\rvert = \lvert -2 \rvert^2 = (-2)^2 = 4$

NOTE It is important to recognize the difference between expressions such as $(-2)^4$ and -2^4. In $(-2)^4$, the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. Hence, $(-2)^4 = 16$, whereas $-2^4 = -16$.

Study Tip

Rarely in algebra is there only one way to solve a problem. Don't be concerned if the steps you use to solve a problem are not exactly the same as the steps presented in this text. The important thing is to use steps that you understand *and,* of course, are justified by the rules of algebra. For instance, you might prefer the following steps for Example 2(d).

$$\left(\frac{3x^2}{y}\right)^{-2} = \left(\frac{y}{3x^2}\right)^2 = \frac{y^2}{9x^4}$$

Note how Property 3 is used in the first step of this solution. The fractional form of this property is

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.$$

The properties of exponents listed on the previous page apply to *all* integers m and n, not just positive integers. For instance, by Property 2, you can write

$$\frac{3^4}{3^{-5}} = 3^{4-(-5)} = 3^{4+5} = 3^9.$$

EXAMPLE 1 *Using Properties of Exponents*

a. $(-3ab^4)(4ab^{-3}) = -12(a)(a)(b^4)(b^{-3}) = -12a^2b$

b. $(2xy^2)^3 = 2^3(x)^3(y^2)^3 = 8x^3y^6$

c. $3a(-4a^2)^0 = 3a(1) = 3a, \quad a \neq 0$

d. $\left(\frac{5x^3}{y}\right)^2 = \frac{5^2(x^3)^2}{y^2} = \frac{25x^6}{y^2}$

EXAMPLE 2 *Rewriting with Positive Exponents*

a. $x^{-1} = \frac{1}{x}$ Property 3: $a^{-n} = \frac{1}{a^n}$

b. $\frac{1}{3x^{-2}} = \frac{1(x^2)}{3} = \frac{x^2}{3}$ -2 exponent does not apply to 3.

c. $\frac{12a^3b^{-4}}{4a^{-2}b} = \frac{12a^3 \cdot a^2}{4b \cdot b^4} = \frac{3a^5}{b^5}$

d. $\left(\frac{3x^2}{y}\right)^{-2} = \frac{3^{-2}(x^2)^{-2}}{y^{-2}} = \frac{3^{-2}x^{-4}}{y^{-2}} = \frac{y^2}{3^2x^4} = \frac{y^2}{9x^4}$

e. $\frac{x^{-2}}{y^{-2}} = \frac{y^2}{x^2}$

EXAMPLE 3 *Calculators and Exponents*

Expression	*Graphing Calculator Keystrokes*	*Display*
a. $13^4 + 5$	13 $\boxed{\wedge}$ 4 $\boxed{+}$ 5 $\boxed{\text{ENTER}}$	28566
b. $3^{-2} + 4^{-1}$	3 $\boxed{\wedge}$ $\boxed{}$ 2 $\boxed{+}$ 4 $\boxed{\wedge}$ $\boxed{}$ 1 $\boxed{\text{ENTER}}$	.3611111111
c. $\frac{3^5 + 1}{3^5 - 1}$	$\boxed{(}$ 3 $\boxed{\wedge}$ 5 $\boxed{+}$ 1 $\boxed{)}$ $\boxed{\div}$ $\boxed{(}$ 3 $\boxed{\wedge}$ 5 $\boxed{-}$ 1 $\boxed{)}$ $\boxed{\text{ENTER}}$	1.008264463

TECHNOLOGY

The calculator keystrokes in Example 3 and throughout this text are for the *TI-83* or the *TI-82* graphing calculator. The corresponding scientific calculator keystrokes are given in the appendix.

Scientific Notation

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, a drop of water contains more than 33 billion billion molecules—that is, 33 followed by 18 zeros.

33,000,000,000,000,000,000

It is convenient to write such numbers in **scientific notation.** This notation has the form $\pm c \times 10^n$, where $1 \le c < 10$ and n is an integer. Thus, the number of molecules in a drop of water can be written in scientific notation as

$$3.3 \times 10{,}000{,}000{,}000{,}000{,}000{,}000 = 3.3 \times 10^{19}.$$

The *positive* exponent 19 indicates that the number is *large* (10 or more) and that the decimal point has been moved 19 places. A *negative* exponent indicates that the number is *small* (less than 1). For instance, the mass (in grams) of one electron is approximately

$$9.0 \times 10^{-28} = 0.\underbrace{0000000000000000000000000009}_{\text{28 decimal places}}.$$

EXAMPLE 4 *Scientific Notation*

a. $1.345 \times 10^2 = 134.5$

b. $0.0000782 = 7.82 \times 10^{-5}$

c. $9.36 \times 10^{-6} = 0.00000936$

d. $836{,}100{,}000 = 8.361 \times 10^8$

NOTE Most calculators switch to scientific notation when they are showing large (or small) numbers that exceed the display range. Try evaluating $86{,}500{,}000 \times 6000$. If your calculator follows standard conventions, its display should be

$\boxed{5.19 \quad 11}$ or $\boxed{5.19 \text{ E } 11}$

which is 5.19×10^{11}.

EXAMPLE 5 *Using Scientific Notation with a Calculator*

Use a calculator to evaluate $65{,}000 \times 3{,}400{,}000{,}000$.

Solution

Because $65{,}000 = 6.5 \times 10^4$ and $3{,}400{,}000{,}000 = 3.4 \times 10^9$, you can multiply the two numbers using the following graphing calculator steps.

6.5 $\boxed{\text{EE}}$ 4 $\boxed{\times}$ 3.4 $\boxed{\text{EE}}$ 9 $\boxed{\text{ENTER}}$

After entering these keystrokes, the calculator display should read $\boxed{2.21 \text{ E } 14}$. Therefore, the product of the two numbers is

$$\begin{aligned}(6.5 \times 10^4)(3.4 \times 10^9) &= 2.21 \times 10^{14}\\ &= 221{,}000{,}000{,}000{,}000.\end{aligned}$$

Radicals and Their Properties

A **square root** of a number is one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of 25. In a similar way, a **cube root** of a number is one of its three equal factors.

Definition of nth Root of a Number

Let a and b be real numbers and let $n \geq 2$ be a positive integer. If

$$a = b^n$$

then b is an ***n*th root of *a***. In $n = 2$, the root is a **square root.** If $n = 3$, the root is a **cube root.**

Some numbers have more than one nth root. For example, both 5 and -5 are square roots of 25. The **principal *n*th root** of a number is defined as follows.

Principal nth Root of a Number

Let a be a real number that has at least one nth root. The **principal *n*th root of *a*** is the nth root that has the same sign as a. It is denoted by a **radical symbol**

$$\sqrt[n]{a}. \qquad \text{Principal } n\text{th root}$$

The positive integer n is the **index** of the radical, and the number a is the **radicand.** If $n = 2$, we omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$. (The plural of index is *indices.*)

EXAMPLE 6 ***Evaluating Expressions Involving Radicals***

a. $\sqrt{36} = 6$ because $6^2 = 36$.

b. $-\sqrt{36} = -6$ because $6^2 = 36$.

c. $\sqrt[3]{\dfrac{125}{64}} = \dfrac{5}{4}$ because $\left(\dfrac{5}{4}\right)^3 = \dfrac{5^3}{4^3} = \dfrac{125}{64}$.

d. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

e. $\sqrt[4]{-81}$ is not a real number because there is no real number that can be raised to the fourth power to produce -81.

Here are some generalizations about the nth roots of a real number.

1. If a is a positive real number and n is a positive *even* integer, then a has exactly two real nth roots denoted by $\sqrt[n]{a}$ and $-\sqrt[n]{a}$. See Examples 6(a) and 6(b).
2. If a is any real number and n is an *odd* integer, then a has only one real nth root denoted by $\sqrt[n]{a}$. See Examples 6(c) and 6(d).
3. If a is a negative real number and n is an *even* integer, then a has no real nth root. See Example 6(e).
4. $\sqrt[n]{0} = 0$.

Integers such as 1, 4, 9, 16, 25, and 36 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers.

Property	*Example*
1. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
3. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}, \quad b \neq 0$	$\dfrac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\dfrac{27}{9}} = \sqrt[4]{3}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$
5. $\left(\sqrt[n]{a}\right)^n = a$	$\left(\sqrt{3}\right)^2 = 3$
6. For n even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\sqrt{(-12)^2} = \lvert -12 \rvert = 12$
For n odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{(-12)^3} = -12$

NOTE A common special case of Property 6 is $\sqrt{a^2} = \lvert a \rvert$.

EXAMPLE 7 ***Using Properties of Radicals***

a. $\sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$

b. $\left(\sqrt[3]{5}\right)^3 = 5$

c. $\sqrt[3]{x^3} = x$

Simplifying Radicals

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (accomplished by a process called *rationalizing the denominator*).
3. The index of the radical is reduced.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical, and the "leftover" factors make up the new radicand.

EXAMPLE 8 *Simplifying Even Roots*

a. $\sqrt[4]{48} = \sqrt[4]{\underbrace{16}_{\text{Perfect 4th power}} \cdot \underbrace{3}_{\text{Leftover factor}}} = \sqrt[4]{2^4 \cdot 3} = 2\sqrt[4]{3}$

b. $\sqrt{75x^3} = \sqrt{\underbrace{25x^2}_{\text{Perfect square}} \cdot \underbrace{3x}_{\text{Leftover factor}}}$ Find largest square factor.

$= \sqrt{(5x)^2 \cdot 3x}$

$= 5x\sqrt{3x}$ Find root of perfect square.

c. $\sqrt[4]{(5x)^4} = |5x| = 5|x|$

NOTE In Example 8(b), the expression $\sqrt{75x^3}$ makes sense only for nonnegative values of x.

EXAMPLE 9 *Simplifying Odd Roots*

a. $\sqrt[3]{24} = \sqrt[3]{\underbrace{8}_{\text{Perfect cube}} \cdot \underbrace{3}_{\text{Leftover factor}}} = \sqrt[3]{2^3 \cdot 3} = 2\sqrt[3]{3}$

b. $\sqrt[3]{24a^4} = \sqrt[3]{\underbrace{8a^3}_{\text{Perfect cube}} \cdot \underbrace{3a}_{\text{Leftover factor}}}$ Find largest cube factor.

$= \sqrt[3]{(2a)^3 \cdot 3a}$

$= 2a\sqrt[3]{3a}$ Find root of perfect cube.

c. $\sqrt[3]{-40x^6} = \sqrt[3]{(-8x^6) \cdot 5} = \sqrt[3]{(-2x^2)^3 \cdot 5} = -2x^2\sqrt[3]{5}$

Rationalizing Denominators and Numerators

To rationalize a denominator or numerator of the form $a - b\sqrt{m}$ or $a + b\sqrt{m}$, multiply both numerator and denominator by a **conjugate:** $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are conjugates of each other. If $a = 0$, then the rationalizing factor for $\sqrt{m}$ is itself, $\sqrt{m}$.

EXAMPLE 10 *Rationalizing Single-Term Denominators*

a. $\dfrac{5}{2\sqrt{3}} = \dfrac{5}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{5\sqrt{3}}{2(3)} = \dfrac{5\sqrt{3}}{6}$

b. $\dfrac{2}{\sqrt[3]{5}} = \dfrac{2}{\sqrt[3]{5}} \cdot \dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} = \dfrac{2\sqrt[3]{5^2}}{\sqrt[3]{5^3}} = \dfrac{2\sqrt[3]{25}}{5}$

EXAMPLE 11 *Rationalizing a Denominator with Two Terms*

$\dfrac{2}{3+\sqrt{7}} = \dfrac{2}{3+\sqrt{7}} \cdot \dfrac{3-\sqrt{7}}{3-\sqrt{7}}$ Multiply numerator and denominator by conjugate.

$= \dfrac{2(3-\sqrt{7})}{(3)^2 - (\sqrt{7})^2}$

$= \dfrac{2(3-\sqrt{7})}{9-7}$

$= \dfrac{2(3-\sqrt{7})}{2}$

$= 3 - \sqrt{7}$ Cancel like factors.

EXAMPLE 12 *Rationalizing the Numerator*

$\dfrac{\sqrt{5}-\sqrt{7}}{2} = \dfrac{\sqrt{5}-\sqrt{7}}{2} \cdot \dfrac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{7}}$ Multiply numerator and denominator by conjugate.

$= \dfrac{5-7}{2(\sqrt{5}+\sqrt{7})}$

$= \dfrac{-2}{2(\sqrt{5}+\sqrt{7})}$

$= \dfrac{-1}{\sqrt{5}+\sqrt{7}}$ Simplify.

NOTE Do not confuse the expression $\sqrt{5} + \sqrt{7}$ with the expression $\sqrt{5+7}$. In general, $\sqrt{x+y}$ does not equal $\sqrt{x} + \sqrt{y}$. Similarly, $\sqrt{x^2+y^2}$ does not equal $x + y$.

Rational Exponents

DEFINITION OF RATIONAL EXPONENTS

If a is a real number and n is a positive integer such that the principal nth root of a exists, we define $a^{1/n}$ to be

$$a^{1/n} = \sqrt[n]{a}.$$

Moreover, if m is a positive integer that has no common factor with n, then

$$a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m \quad \text{and} \quad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}.$$

The numerator of a rational exponent denotes the *power* to which the base is raised, and the denominator denotes the *index* or the *root* to be taken, as shown below.

Power

Index

$$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

When you are working with rational exponents, the properties of integer exponents still apply. For instance,

$$2^{1/2}2^{1/3} = 2^{(1/2)+(1/3)} = 2^{5/6}.$$

NOTE Rational exponents can be tricky, and you must remember that the expression $b^{m/n}$ is not defined unless $\sqrt[n]{b}$ is a real number. This restriction produces some unusual-looking results. For instance, the number $(-8)^{1/3}$ is defined because $\sqrt[3]{-8} = -2$, but the number $(-8)^{2/6}$ is undefined because $\sqrt[6]{-8}$ is not a real number.

EXAMPLE 13 *Changing from Radical to Exponential Form*

a. $\sqrt{3} = 3^{1/2}$

b. $\sqrt{(3xy)^5} = \sqrt[2]{(3xy)^5} = (3xy)^{(5/2)}$

c. $2x\sqrt[4]{x^3} = (2x)(x^{3/4}) = 2x^{1+(3/4)} = 2x^{7/4}$

EXAMPLE 14 *Changing from Exponential to Radical Form*

a. $(x^2 + y^2)^{3/2} = \left(\sqrt{x^2 + y^2}\right)^3 = \sqrt{(x^2 + y^2)^3}$

b. $2y^{3/4}z^{1/4} = 2(y^3z)^{1/4} = 2\sqrt[4]{y^3z}$

c. $a^{-3/2} = \dfrac{1}{a^{3/2}} = \dfrac{1}{\sqrt{a^3}}$

d. $x^{0.2} = x^{1/5} = \sqrt[5]{x}$

Rational exponents are particularly useful for evaluating roots of numbers on a calculator, for reducing the index of a radical, and for simplifying expressions encountered in calculus.

The *Interactive* CD-ROM shows every example with its solution; clicking on the *Try It!* button brings up similar problems. Guided Examples and Integrated Examples show step-by-step solutions to additional examples. Integrated Examples are related to several concepts in the section.

EXAMPLE 15 *Simplifying with Rational Exponents*

a. $(27)^{2/6} = (27)^{1/3} = \sqrt[3]{27} = 3$

b. $(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

c. $(-5x^{5/3})(3x^{-3/4}) = -15x^{(5/3)-(3/4)} = -15x^{11/12}, \quad x \neq 0$

d. $\sqrt[9]{a^3} = a^{3/9} = a^{1/3} = \sqrt[3]{a}$

e. $\sqrt[3]{\sqrt{125}} = \sqrt[6]{125} = \sqrt[6]{(5)^3} = 5^{3/6} = 5^{1/2} = \sqrt{5}$

f. $(2x - 1)^{4/3}(2x - 1)^{-1/3} = (2x - 1)^{(4/3)-(1/3)}$

$$= 2x - 1, \quad x \neq \frac{1}{2}$$

g. $\dfrac{x - 1}{(x - 1)^{-1/2}} = \dfrac{x - 1}{(x - 1)^{-1/2}} \cdot \dfrac{\sqrt{x - 1}}{\sqrt{x - 1}} = \dfrac{(x - 1)^{3/2}}{(x - 1)^0}$

$$= (x - 1)^{3/2}, \quad x \neq 1$$

Radical expressions can be combined (added or subtracted) if they are **like radicals**—that is, if they have the same index and radicand. For instance, $\sqrt{2}$, $3\sqrt{2}$, and $\frac{1}{2}\sqrt{2}$ are like radicals, but $\sqrt{3}$ and $\sqrt{2}$ are unlike radicals. To determine whether two radicals are like radicals, you should first simplify each radical.

EXAMPLE 16 *Combining Radicals*

a. $2\sqrt{48} - 3\sqrt{27} = 2\sqrt{16 \cdot 3} - 3\sqrt{9 \cdot 3}$ Find square factors.

$= 8\sqrt{3} - 9\sqrt{3}$ Find square roots.

$= (8 - 9)\sqrt{3}$ Combine like terms.

$= -\sqrt{3}$

b. $\sqrt[3]{16x} - \sqrt[3]{54x^4} = \sqrt[3]{8 \cdot 2x} - \sqrt[3]{27 \cdot x^3 \cdot 2x}$

$= 2\sqrt[3]{2x} - 3x\sqrt[3]{2x}$

$= (2 - 3x)\sqrt[3]{2x}$

Radicals and Calculators

There are four methods of evaluating radicals on most graphing calculators. For square roots, you can use the *square root key* $\boxed{\sqrt{\ }}$. For cube roots, you can use the *cube root key* $\boxed{\sqrt[3]{\ }}$. For other roots, you can first convert the radical to exponential form and then use the *exponential key* $\boxed{\wedge}$, or you can use the *nth root key* $\boxed{\sqrt[x]{\ }}$.

EXAMPLE 17 ***Evaluating Radicals with a Calculator***

Use a calculator to evaluate $\sqrt[4]{56} = 56^{1/4}$.

Graphing Calculator Keystrokes

a. 56 $\boxed{\wedge}$ $\boxed{(}$ 1 $\boxed{\div}$ 4 $\boxed{)}$ $\boxed{\text{ENTER}}$

b. 4 $\boxed{\text{MATH}}$ (MATH) (5: $\sqrt[x]{\ }$) 56 $\boxed{\text{ENTER}}$

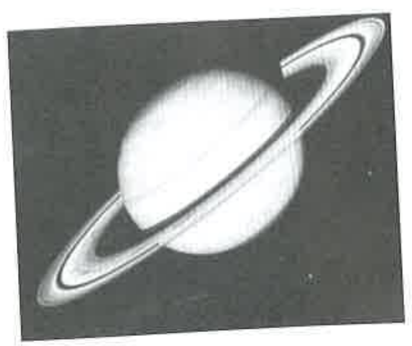
For each of these two keystroke sequences, the display is $\sqrt[4]{56} \approx 2.7355648$.

GROUP ACTIVITY

A FAMOUS MATHEMATICAL DISCOVERY

Johannes Kepler (1571–1630), a well-known German astronomer, discovered a relationship between the average distance of a planet from the sun and the time (or period) it takes the planet to orbit the sun. People then knew that planets that are closer to the sun take less time to complete an orbit than planets that are farther from the sun. Kepler discovered that the distance and period are related by an exact mathematical formula. The table shows the average distance x (in astronomical units) and period y (in years) for the six planets that are closest to the sun. By completing the table, can you rediscover Kepler's relationship? Discuss your conclusions.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
x	0.387	0.723	1.0	1.523	5.203	9.541
$\sqrt{x}$						
y	0.241	0.615	1.0	1.881	11.861	29.457
$\sqrt[3]{y}$						



Saturn, 886 million miles from the sun, completes one rotation every 10.4 earth hours and orbits the sun once in 29.46 earth years. *(Photo: NASA)*

The *Interactive* CD-ROM provides additional help with Warm-Up exercises by providing a hypertext link to the section in which the concept was introduced.

WARM UP

Place the correct inequality symbol (< or >) between the two numbers.

1. $-4 \ \square\ -2$ **2.** $0 \ \square\ -3$

3. $\sqrt{3} \ \square\ 1.73$ **4.** $-\pi \ \square\ -3$

Find the distance between the two numbers.

5. $-6, 4$ **6.** $-2, 10$

Evaluate the expression.

7. $|-7| + |7|$ **8.** $-|8 - 10|$

9. $18 + \frac{25 - 13}{4}$ **10.** $\frac{7}{8} - \left(\frac{4}{5} \div \frac{8}{25}\right)$

The *Interactive* CD-ROM contains step-by-step solutions to all odd-numbered Section and Review Exercises. It also provides Tutorial Exercises, which link to Guided Examples for additional help.

P.2 Exercises

In Exercises 1 and 2, write the expression as a repeated multiplication problem.

1. -0.4^6

2. $(-2)^7$

In Exercises 3 and 4, write the expression using exponential notation.

3. $(-10)(-10)(-10)(-10)(-10)$

4. $-\left(\frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2}\right)$

In Exercises 5–10, evaluate the expression.

5. (a) $4^2 \cdot 3$ (b) $3 \cdot 3^3$

6. (a) $\frac{5^5}{5^2}$ (b) $\frac{3^2}{3^4}$

7. (a) $(3^3)^2$ (b) -3^2

8. (a) $(2^3 \cdot 3^2)^2$ (b) $\left(-\frac{3}{5}\right)^3\left(\frac{5}{3}\right)^2$

9. (a) $\frac{3}{3^{-4}}$ (b) $24(-2)^{-5}$

10. (a) $\frac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}$ (b) $(-2)^0$

In Exercises 11–14, use a calculator to evaluate the expression. (Round to three decimal places.)

11. $(-4)^3(5^2)$ **12.** $(8^{-4})(10^3)$

13. $\frac{3^6}{7^3}$ **14.** $\frac{4^3}{3^{-4}}$

In Exercises 15–18, evaluate the expression for the value of x.

	Expression	*Value*
15.	$-3x^3$	2
16.	$7x^{-2}$	4
17.	$6x^0 - (6x)^0$	10
18.	$5(-x)^3$	3

In Exercises 19–32, simplify the expression.

19. (a) $(-5z)^3$ (b) $5x^4(x^2)$
20. (a) $(3x)^2$ (b) $(4x^3)^2$
21. (a) $6y^2(2y^4)^2$ (b) $\dfrac{3x^5}{x^3}$
22. (a) $(-z)^3(3z^4)$ (b) $\dfrac{25y^8}{10y^4}$
23. (a) $\dfrac{7x^2}{x^3}$ (b) $\dfrac{12(x+y)^3}{9(x+y)}$
24. (a) $\dfrac{r^4}{r^6}$ (b) $\left(\dfrac{4}{y}\right)^3\left(\dfrac{3}{y}\right)^4$
25. (a) $(x+5)^0, \quad x \neq -5$ (b) $(2x^2)^{-2}$
26. (a) $(2x^5)^0, \quad x \neq 0$ (b) $(z+2)^{-3}(z+2)^{-1}$
27. (a) $(-2x^2)^3(4x^3)^{-1}$ (b) $\left(\dfrac{x}{10}\right)^{-1}$
28. (a) $(4y^{-2})(8y^4)$ (b) $\left(\dfrac{x^{-3}y^4}{5}\right)^{-3}$
29. (a) $(4a^{-2}b^3)^{-3}$ (b) $\left(\dfrac{5x^2}{y^{-2}}\right)^{-4}$
30. (a) $[(x^2y^{-2})^{-1}]^{-1}$ (b) $(5x^2z^6)^3(5x^2z^6)^{-3}$
31. (a) $3^n \cdot 3^{2n}$ (b) $\left(\dfrac{a^{-2}}{b^{-2}}\right)\left(\dfrac{b}{a}\right)^3$
32. (a) $\dfrac{x^2 \cdot x^n}{x^3 \cdot x^n}$ (b) $\left(\dfrac{a^{-3}}{b^{-3}}\right)\left(\dfrac{a}{b}\right)^3$

In Exercises 33–44, fill in the missing description.

	Radical Form	*Rational Exponent Form*
33.	$\sqrt{9} = 3$	
34.	$\sqrt[3]{64} = 4$	
35.		$32^{1/5} = 2$
36.		$-(144^{1/2}) = -12$
37.		$196^{1/2} = 14$
38.	$\sqrt[3]{614.125} = 8.5$	
39.	$\sqrt[3]{-216} = -6$	
40.		$(-243)^{1/5} = -3$
41.		$27^{2/3} = 9$
42.	$\left(\sqrt[4]{81}\right)^3 = 27$	
43.	$\sqrt[4]{81^3} = 27$	
44.		$16^{5/4} = 32$

In Exercises 45–54, evaluate each expression. (Do not use a calculator.)

45. (a) $\sqrt{9}$ (b) $\sqrt[3]{8}$
46. (a) $\sqrt{49}$ (b) $\sqrt[3]{\frac{27}{8}}$
47. (a) $-\sqrt[3]{-27}$ (b) $\dfrac{4}{\sqrt{64}}$
48. (a) $\sqrt[3]{0}$ (b) $\dfrac{\sqrt[4]{81}}{3}$
49. (a) $\left(\sqrt[3]{-125}\right)^3$ (b) $27^{1/3}$
50. (a) $\sqrt[4]{562^4}$ (b) $36^{3/2}$
51. (a) $32^{-3/5}$ (b) $\left(\frac{16}{81}\right)^{-3/4}$
52. (a) $100^{-3/2}$ (b) $\left(\frac{9}{4}\right)^{-1/2}$
53. (a) $\left(-\dfrac{1}{64}\right)^{-1/3}$ (b) $\left(\dfrac{1}{\sqrt{32}}\right)^{-2/5}$
54. (a) $\left(-\dfrac{125}{27}\right)^{-1/3}$ (b) $-\left(\dfrac{1}{125}\right)^{-4/3}$

In Exercises 55–58, use a calculator to approximate the number. (Round to three decimal places.)

55. (a) $\sqrt{57}$ (b) $\sqrt[5]{-27^3}$
56. (a) $\sqrt[3]{45^2}$ (b) $\sqrt[6]{125}$
57. (a) $(1.2^{-2})\sqrt{75} + 3\sqrt{8}$ (b) $\dfrac{-3+\sqrt{21}}{3}$
58. (a) $(15.25)^{-1.4}$ (b) $(3.4)^{2.5}$

In Exercises 59–64, simplify by removing all possible factors from the radical.

59. (a) $\sqrt{8}$ (b) $\sqrt[3]{24}$
60. (a) $\sqrt[3]{\frac{16}{27}}$ (b) $\sqrt{\frac{75}{4}}$
61. (a) $\sqrt{72x^3}$ (b) $\sqrt{\dfrac{18^2}{z^3}}$
62. (a) $\sqrt{54xy^4}$ (b) $\sqrt{\dfrac{32a^4}{b^2}}$
63. (a) $\sqrt[3]{16x^5}$ (b) $\sqrt{75x^2y^{-4}}$
64. (a) $\sqrt[4]{(3x^2)^4}$ (b) $\sqrt[5]{96x^5}$

In Exercises 65–70, perform the operations and simplify.

65. $5^{4/3} \cdot 5^{8/3}$
66. $\dfrac{8^{12/5}}{8^{2/5}}$
67. $\dfrac{(2x^2)^{3/2}}{2^{1/2}x^4}$
68. $\dfrac{x^{4/3}y^{2/3}}{(xy)^{1/3}}$
69. $\dfrac{x^{-3} \cdot x^{1/2}}{x^{3/2} \cdot x^{-1}}$
70. $\dfrac{5^{-1/2} \cdot 5x^{5/2}}{(5x)^{3/2}}$

In Exercises 71–74, rationalize the denominator. Then simplify your answer.

71. (a) $\dfrac{1}{\sqrt{3}}$ (b) $\dfrac{8}{\sqrt[3]{2}}$
72. (a) $\dfrac{5}{\sqrt{10}}$ (b) $\dfrac{5}{\sqrt[3]{(5x)^2}}$
73. (a) $\dfrac{2x}{5-\sqrt{3}}$ (b) $\dfrac{3}{\sqrt{5}+\sqrt{6}}$
74. (a) $\dfrac{5}{\sqrt{14}-2}$ (b) $\dfrac{5}{2\sqrt{10}-5}$

In Exercises 75–78, rationalize the numerator. Then simplify your answer.

75. (a) $\dfrac{\sqrt{8}}{2}$ (b) $\sqrt[3]{\dfrac{9}{25}}$
76. (a) $\dfrac{\sqrt{2}}{3}$ (b) $\sqrt[4]{\dfrac{5}{4}}$
77. (a) $\dfrac{\sqrt{5}+\sqrt{3}}{3}$ (b) $\dfrac{\sqrt{7}-3}{4}$
78. (a) $\dfrac{\sqrt{3}-\sqrt{2}}{2}$ (b) $\dfrac{2\sqrt{3}+\sqrt{3}}{3}$

In Exercises 79 and 80, reduce the index of the radical.

79. (a) $\sqrt[4]{3^2}$ (b) $\sqrt[6]{(x+1)^4}$
80. (a) $\sqrt[6]{x^3}$ (b) $\sqrt[4]{(3x^2)^4}$

In Exercises 81 and 82, write as a single radical. Then simplify your answer.

81. (a) $\sqrt{\sqrt{32}}$ (b) $\sqrt{\sqrt[4]{2x}}$
82. (a) $\sqrt{\sqrt{243(x+1)}}$ (b) $\sqrt{\sqrt[3]{10a^7b}}$

In Exercises 83–86, simplify the expression.

83. (a) $2\sqrt{50} + 12\sqrt{8}$ (b) $10\sqrt{32} - 6\sqrt{18}$
84. (a) $4\sqrt{27} - \sqrt{75}$ (b) $\sqrt[3]{16} + 3\sqrt[3]{54}$
85. (a) $5\sqrt{x} - 3\sqrt{x}$ (b) $-2\sqrt{9y} + 10\sqrt{y}$
86. (a) $3\sqrt{x+1} + 10\sqrt{x+1}$
(b) $7\sqrt{80x} - 2\sqrt{125x}$

In Exercises 87–90, fill in the blank with <, =, or >.

87. $\sqrt{5} + \sqrt{3} \quad \square \quad \sqrt{5+3}$
88. $\sqrt{\dfrac{3}{11}} \quad \square \quad \dfrac{\sqrt{3}}{\sqrt{11}}$
89. $5 \quad \square \quad \sqrt{3^2+2^2}$
90. $5 \quad \square \quad \sqrt{3^2+4^2}$

In Exercises 91–94, write the number in scientific notation.

91. Land Area of Earth: 57,500,000 square miles
92. Light Year: 9,461,000,000,000,000 kilometers
93. Relative Density of Hydrogen: 0.0000899 gram per cm^3
94. One Micron (Millionth of Meter): 0.00003937 inch

In Exercises 95–98, write the number in decimal form.

95. U.S. Daily Coca-Cola Consumption: 5.24×10^8 servings

96. Interior Temperature of Sun: 1.3×10^7 degrees Celsius

97. Charge of Electron: 4.8×10^{-10} electrostatic unit

98. Width of Human Hair: 9.0×10^{-4} meter

In Exercises 99–102, use a calculator to evaluate the expression. (Round to three decimal places.)

99. (a) $750\left(1 + \dfrac{0.11}{365}\right)^{800}$

(b) $\dfrac{67{,}000{,}000 + 93{,}000{,}000}{0.0052}$

100. (a) $(9.3 \times 10^6)^3(6.1 \times 10^{-4})$

(b) $\dfrac{(2.414 \times 10^4)^6}{(1.68 \times 10^5)^5}$

101. (a) $\sqrt{4.5 \times 10^9}$ (b) $\sqrt[3]{6.3 \times 10^4}$

102. (a) $(2.65 \times 10^{-4})^{1/3}$ (b) $\sqrt{9 \times 10^{-4}}$

103. ***Exploration*** List all possible unit digits of the square of a positive integer. Use that list to determine whether $\sqrt{5233}$ is an integer.

104. ***Think About It*** Square the real number $2/\sqrt{5}$ and note that the radical is eliminated from the denominator. Is this equivalent to rationalizing the denominator? Why or why not?

105. ***Period of a Pendulum*** The period T in seconds of a pendulum is

$$T = 2\pi\sqrt{\frac{L}{32}}$$

where L is the length of the pendulum in feet. Find the period of a pendulum whose length is 2 feet.

106. ***Mathematical Modeling*** A funnel is filled with water to a height of h centimeters. The time t (in seconds) for the funnel to empty is

$$t = 0.03[12^{5/2} - (12 - h)^{5/2}], \quad 0 \le h \le 12.$$

Find t for $h = 7$ centimeters.

107. ***Declining Balances Depreciation*** Find the annual depreciation rate r for the bar graph below. To find the annual depreciation rate by the **declining balances method,** use the formula

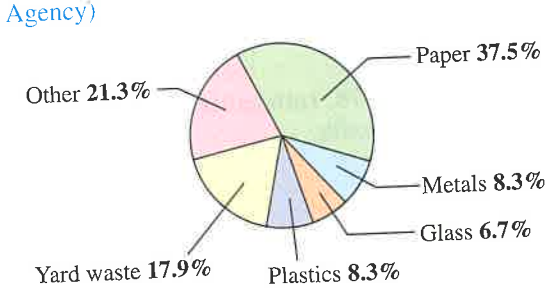
$$r = 1 - \left(\frac{S}{C}\right)^{1/n}$$

where n is the useful life of the item (in years), S is the salvage value (in dollars), and C is the original cost (in dollars).

108. ***Erosion*** A stream of water moving at the rate of v feet per second can carry particles of size $0.03\sqrt{v}$ inches. Find the size particle that can be carried by a stream flowing at the rate of $\frac{3}{4}$ foot per second.

109. ***Speed of Light*** The speed of light is 11,160,000 miles per minute. The distance from the sun to the earth is 93,000,000 miles. Find the time for light to travel from the sun to the earth.

110. ***Organizing Data*** There were 1.957×10^8 tons of municipal waste generated in the U.S. in 1990. Find the number of tons for each of the categories in the figure. (Source: U.S. Environmental Protection Agency)



P.3 Polynomials and Factoring

See Exercise 61 on page 37 for an example of how a polynomial can be used to model the total stopping distance of an automobile.

Polynomials ▫ Operations with Polynomials ▫ Special Products ▫ Factoring ▫ Factoring Special Polynomial Forms ▫ Trinomials with Binomial Factors ▫ Factoring by Grouping

Polynomials

An **algebraic expression** is a collection of letters called **variables** and real numbers organized in some manner by addition, subtraction, multiplication, or division. The most common type of algebraic expression is the **polynomial.** Some examples are

$$2x + 5, \quad 3x^4 - 7x^2 + 2x + 4, \quad \text{and} \quad 5x^2y^2 - xy + 3.$$

The first two are *polynomials in x* and the third is a *polynomial in x and y.* The terms of a polynomial in x have the form ax^k, where a is the **coefficient** and k is the **degree** of the term. For instance, the third-degree polynomial

$$2x^3 - 5x^2 + 1 = 2x^3 + (-5)x^2 + (0)x + 1$$

has coefficients 2, −5, 0, and 1.

NOTE Polynomials with one, two, and three terms are called **monomials, binomials,** and **trinomials,** respectively.

DEFINITION OF A POLYNOMIAL IN X

Let $a_0, a_1, a_2, \ldots, a_n$ be *real numbers* and let n be a *nonnegative integer.* A **polynomial in x** is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where $a_n \neq 0$. The polynomial is of **degree** n, a_n is the **leading coefficient,** and a_0 is the **constant term.**

In **standard form,** a polynomial is written with descending powers of x.

NOTE A polynomial that has all zero coefficients is called the **zero polynomial,** denoted by 0. No degree is assigned to this particular polynomial. For polynomials in more than one variable, the degree of a *term* is the sum of the exponents of the variables in the term. The degree of the *polynomial* is the highest degree of its terms.

EXAMPLE 1 Writing Polynomials in Standard Form

	Polynomial	*Standard Form*	*Degree*
a.	$4x^2 - 5x^7 - 2 + 3x$	$-5x^7 + 4x^2 + 3x - 2$	7
b.	$4 - 9x^2$	$-9x^2 + 4$	2
c.	8	$8 \; (8 = 8x^0)$	0

Operations with Polynomials

You can **add** and **subtract** polynomials in much the same way you add and subtract real numbers. Simply add or subtract the *like terms* (terms having the same variables to the same powers) by adding their coefficients. For instance, $-3xy^2$ and $5xy^2$ are like terms, and their sum is

$$-3xy^2 + 5xy^2 = (-3 + 5)xy^2 = 2xy^2.$$

EXAMPLE 2 ***Sums and Differences of Polynomials***

a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$

$= (5x^3 + x^3) + (2x^2 - 7x^2) - x + (8 - 3)$ — Group like terms.

$= 6x^3 - 5x^2 - x + 5$ — Combine like terms.

b. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$

$= 7x^4 - x^2 - 4x + 2 - 3x^4 + 4x^2 - 3x$

$= (7x^4 - 3x^4) + (4x^2 - x^2) + (-3x - 4x) + 2$ — Group like terms.

$= 4x^4 + 3x^2 - 7x + 2$ — Combine like terms.

NOTE A common mistake is to fail to change the sign of *each* term inside parentheses preceded by a negative sign. For instance, note that

$$-(x^2 - x + 3) = -x^2 + x - 3$$

and

$$-(x^2 - x + 3) \neq -x^2 - x + 3.$$

To find the **product** of two polynomials, use the left and right Distributive Properties.

EXAMPLE 3 ***Multiplying Polynomials: The FOIL Method***

Multiply $(3x - 2)$ by $(5x + 7)$.

Solution

$$\begin{aligned}(3x - 2)(5x + 7) &= 3x(5x + 7) - 2(5x + 7)\\ &= (3x)(5x) + (3x)(7) - (2)(5x) - (2)(7)\\ &= 15x^2 + 21x - 10x - 14\end{aligned}$$

Product of First terms | Product of Outer terms | Product of Inner terms | Product of Last terms

$$= 15x^2 + 11x - 14$$

Note that in this **FOIL Method** for binomials the outer (O) and inner (I) terms are alike and can be combined into one term.

Special Products

SPECIAL PRODUCTS

Let u and v be real numbers, variables, or algebraic expressions.

Special Product	*Example*
Sum and Difference of Same Terms	
$(u + v)(u - v) = u^2 - v^2$	$(x + 4)(x - 4) = x^2 - 4^2 = x^2 - 16$
Square of a Binomial	
$(u + v)^2 = u^2 + 2uv + v^2$	$(x + 3)^2 = x^2 + 2(x)(3) + 3^2 = x^2 + 6x + 9$
$(u - v)^2 = u^2 - 2uv + v^2$	$(3x - 2)^2 = (3x)^2 - 2(3x)(2) + 2^2 = 9x^2 - 12x + 4$
Cube of a Binomial	
$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$	$(x + 2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3 = x^3 + 6x^2 + 12x + 8$
$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$	$(x - 1)^3 = x^3 - 3x^2(1) + 3x(1^2) - 1^3 = x^3 - 3x^2 + 3x - 1$

EXAMPLE 4 ***The Product of Two Trinomials***

Find the product of $(x + y - 2)$ and $(x + y + 2)$.

Solution

By grouping $x + y$ in parentheses, you can write

$$\begin{aligned}(x + y - 2)(x + y + 2) &= [(x + y) - 2][(x + y) + 2]\\ &= (x + y)^2 - 2^2\\ &= x^2 + 2xy + y^2 - 4.\end{aligned}$$

TECHNOLOGY

```
Prgm1:EVALUATE
:Lbl 1
:Prompt X
:Disp Y1
:Goto 1
```

There are several ways to use a graphing utility to evaluate a function. Here is one way to evaluate functions on a *TI-82* or a *TI-83*. Begin by entering the program EVALUATE shown at the left. Next, enter the expression $x^2 - 3x + 2$ into Y_1. Then run the program for several values of x. Organize your results in a table. Programs for other graphing calculators may be found in the appendix.

Factoring

The process of writing a polynomial as a product is called **factoring.** It is an important tool for solving equations and for reducing fractional expressions.

Unless noted otherwise, when you are asked to factor a polynomial, you can assume that you are hunting for factors with integer coefficients. If a polynomial cannot be factored using integer coefficients, it is **prime** or **irreducible over the integers.** For instance, the polynomial $x^2 - 3$ is irreducible over the integers. Over the *real numbers,* however, this polynomial can be factored as

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3}).$$

A polynomial is **completely factored** when each of its factors is prime. For instance,

$$x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4)$$

is completely factored, but

$$x^3 - x^2 - 4x + 4 = (x - 1)(x^2 - 4)$$

is not completely factored. Its complete factorization would be

$$x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2).$$

The simplest type of factoring involves a polynomial that can be written as the product of a monomial and another polynomial. The technique used here is the Distributive Property, $a(b + c) = ab + ac$, in the *reverse* direction.

$$ab + ac = a(b + c) \qquad a \text{ is a common factor.}$$

Removing (factoring out) a common factor is the first step in completely factoring a polynomial.

EXAMPLE 5 *Removing Common Factors*

Factor each polynomial.

a. $3x^3 + 9x^2$ **b.** $6x^3 - 4x$ **c.** $(x - 2)(2x) + (x - 2)(3)$

Solution

a. $3x^3 + 9x^2 = 3x^2(x) + 3x^2(3) \qquad 3x^2$ is a common factor.

$\qquad = 3x^2(x + 3)$

b. $6x^3 - 4x = 2x(3x^2) - 2x(2) \qquad 2x$ is a common factor.

$\qquad = 2x(3x^2 - 2)$

c. $(x - 2)(2x) + (x - 2)(3) = (x - 2)(2x + 3) \qquad x - 2$ is a common factor.

Factoring Special Polynomial Forms

FACTORING SPECIAL POLYNOMIAL FORMS

Factored Form	*Example*
Difference of Two Squares	
$u^2 - v^2 = (u + v)(u - v)$	$9x^2 - 4 = (3x)^2 - 2^2 = (3x + 2)(3x - 2)$
Perfect Square Trinomial	
$u^2 + 2uv + v^2 = (u + v)^2$	$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2 = (x + 3)^2$
$u^2 - 2uv + v^2 = (u - v)^2$	$x^2 - 6x + 9 = x^2 - 2(x)(3) + 3^2 = (x - 3)^2$
Sum or Difference of Two Cubes	
$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - 2x + 4)$
$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	$27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)(9x^2 + 3x + 1)$

One of the easiest special polynomial forms to factor is the difference of two squares. Think of the form as follows.

$$\underbrace{u^2 - v^2}_{\text{Difference}} = \underbrace{(u + v)(u - v)}_{\text{Opposite signs}}$$

To recognize perfect square terms, look for coefficients that are squares of integers and variables raised to *even powers.*

NOTE In Example 6, note that the first step in factoring a polynomial is to check for common factors. Once the common factor has been removed, it is often possible to recognize patterns that were not immediately obvious.

EXAMPLE 6 *Removing a Common Factor First*

$$3 - 12x^2 = 3(1 - 4x^2) = 3[1^2 - (2x)^2] = 3(1 + 2x)(1 - 2x)$$

EXAMPLE 7 *Factoring the Difference of Two Squares*

a. $(x + 2)^2 - y^2 = [(x + 2) + y][(x + 2) - y]$

$\qquad = (x + 2 + y)(x + 2 - y)$

b. $16x^4 - 81 = (4x^2)^2 - 9^2$

$\qquad = (4x^2 + 9)(4x^2 - 9) \qquad$ Difference of two squares

$\qquad = (4x^2 + 9)[(2x)^2 - 3^2]$

$\qquad = (4x^2 + 9)(2x + 3)(2x - 3) \qquad$ Difference of two squares

A perfect square trinomial is the square of a binomial, and it has the following form.

$$u^2 + 2uv + v^2 = (u + v)^2 \qquad \text{or} \qquad u^2 - 2uv + v^2 = (u - v)^2$$

Like signs Like signs

Note that the first and last terms are squares and the middle term is twice the product of u and v.

EXAMPLE 8 *Factoring Perfect Square Trinomials*

a. $16x^2 + 8x + 1 = (4x)^2 + 2(4x)(1) + 1^2$
$\qquad = (4x + 1)^2$

b. $x^2 - 10x + 25 = x^2 - 2(x)(5) + 5^2$
$\qquad = (x - 5)^2$

The next two formulas show the sums and differences of cubes. Pay special attention to the signs of the terms.

Like signs Like signs

$$u^3 + v^3 = (u + v)(u^2 - uv + v^2) \qquad u^3 - v^3 = (u - v)(u^2 + uv + v^2)$$

Unlike signs Unlike signs

> **Exploration**
>
> Find a formula for completely factoring $u^6 - v^6$ using the formulas in this section. Use your formula to completely factor $x^6 - 1$ and $x^6 - 64$.

EXAMPLE 9 *Factoring the Difference of Cubes*

$x^3 - 27 = x^3 - 3^3$ Rewrite 27 as 3^3.
$\qquad = (x - 3)(x^2 + 3x + 9)$ Factor.

EXAMPLE 10 *Factoring the Sum of Cubes*

a. $y^3 + 8 = y^3 + 2^3$ Rewrite 8 as 2^3.
$\qquad = (y + 2)(y^2 - 2y + 4)$ Factor.

b. $3(x^3 + 64) = 3(x^3 + 4^3)$ Rewrite 64 as 4^3.
$\qquad = 3(x + 4)(x^2 - 4x + 16)$ Factor.

Trinomials with Binomial Factors

To factor a trinomial of the form $ax^2 + bx + c$, use the following pattern.

Factors of a

$$ax^2 + bx + c = (\square x + \square)(\square x + \square)$$

Factors of c

The goal is to find a combination of factors of a and c so that the outer and inner products add up to the middle term bx. For instance, in the trinomial $6x^2 + 17x + 5$, you can write

F O I L

$$(2x + 5)(3x + 1) = 6x^2 + 2x + 15x + 5 = 6x^2 + 17x + 5.$$

Note that the outer (O) and inner (I) products add up to $17x$.

EXAMPLE 11 *Factoring a Trinomial: Leading Coefficient Is 1*

Factor $x^2 - 7x + 12$.

Solution

The possible factorizations are

$$(x - 2)(x - 6), \quad (x - 1)(x - 12), \quad \text{and} \quad (x - 3)(x - 4).$$

Testing the middle term, you will find the correct factorization to be

$$x^2 - 7x + 12 = (x - 3)(x - 4).$$

EXAMPLE 12 *Factoring a Trinomial: Leading Coefficient Is Not 1*

Factor $2x^2 + x - 15$.

Solution

The eight possible factorizations are as follows.

$(2x - 1)(x + 15) \qquad (2x + 1)(x - 15)$
$(2x - 3)(x + 5) \qquad (2x + 3)(x - 5)$
$(2x - 5)(x + 3) \qquad (2x + 5)(x - 3)$
$(2x - 15)(x + 1) \qquad (2x + 15)(x - 1)$

Testing the middle term, you will find the correct factorization to be

$$2x^2 + x - 15 = (2x - 5)(x + 3).$$

Factoring by Grouping

Sometimes polynomials with more than three terms can be factored by a method called **factoring by grouping.** It is not always obvious which terms to group, and sometimes several different groupings will work.

EXAMPLE 13 *Factoring by Grouping*

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) - (3x - 6) && \text{Group terms.}\\ &= x^2(x-2) - 3(x-2) && \text{Factor groups.}\\ &= (x-2)(x^2-3) && \text{Distributive Property} \end{aligned}$$

Factoring a trinomial can involve quite a bit of trial and error. Some of this trial and error can be lessened by using factoring by grouping.

EXAMPLE 14 *Factoring a Trinomial by Grouping*

Use factoring by grouping to factor $2x^2 + 5x - 3$.

Solution

In the trinomial $2x^2 + 5x - 3$, we have $a = 2$ and $c = -3$, which implies that the product ac is -6. Now, because -6 factors as $(6)(-1)$ and $6 - 1 = 5 = b$, we rewrite the middle term as $5x = 6x - x$. This produces the following.

$$\begin{aligned} 2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 && \text{Rewrite middle term.}\\ &= (2x^2 + 6x) - (x + 3) && \text{Group terms.}\\ &= 2x(x+3) - (x+3) && \text{Factor groups.}\\ &= (x+3)(2x-1) && \text{Distributive Property} \end{aligned}$$

Therefore, the trinomial factors as

$$2x^2 + 5x - 3 = (x+3)(2x-1).$$

GUIDELINES FOR FACTORING POLYNOMIALS

1. Factor out any common factors by the Distributive Property.
2. Factor according to one of the special polynomial forms.
3. Factor as $ax^2 + bx + c = (mx + r)(nx + s)$.
4. Factor by grouping.

GROUP ACTIVITY

A THREE-DIMENSIONAL VIEW OF A SPECIAL PRODUCT

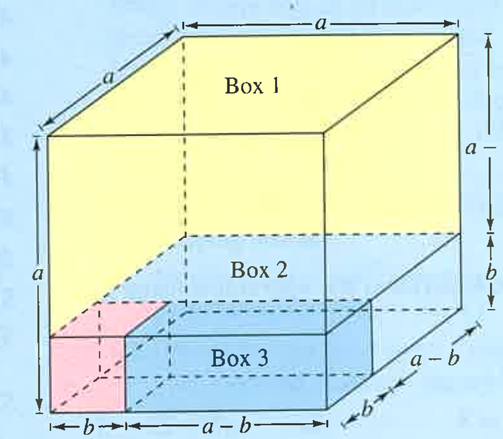
The figure below shows two cubes.

a. The large cube has a volume of a^3.

b. The small cube has a volume of b^3.

If the smaller cube is removed from the larger, the remaining solid has a volume of $a^3 - b^3$ and is composed of three rectangular boxes, labeled Box 1, Box 2, and Box 3. Find the volume of each box and describe how these volumes are related to the special product formula.

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2 + ab + b^2)\\ &= (a-b)a^2 + (a-b)ab + (a-b)b^2 \end{aligned}$$



WARM UP

Simplify the expression.

1. $(7x^2)(6x)$
2. $(10z^3)(-2z^{-1})$
3. $(-3x^2)^3$
4. $(3x^2y^{-1})^0$
5. $\dfrac{27z^5}{12z^2}$
6. $\sqrt{24}\cdot\sqrt{2}$
7. $\left(\dfrac{2x}{3}\right)^{-2}$
8. $\dfrac{4}{\sqrt{8}}$
9. $16^{3/4}$
10. $\sqrt[3]{-27x^3}$

P.3 Exercises

In Exercises 1–14, perform the operations and write the result in standard form.

1. $(6x + 5) - (8x + 15)$
2. $(2x^2 + 1) - (x^2 - 2x + 1)$
3. $-(x^3 - 2) + (4x^3 - 2x)$
4. $-(5x^2 - 1) - (-3x^2 + 5)$
5. $(15x^2 - 6) - (-8x^3 - 14x^2 - 17)$
6. $(15x^4 - 18x - 19) - (13x^4 - 5x + 15)$
7. $5z - [3z - (10z + 8)]$
8. $(y^3 + 1) - [(y^2 + 1) + (3y - 7)]$
9. $3x(x^2 - 2x + 1)$
10. $y^2(4y^2 + 2y - 3)$
11. $-5z(3z - 1)$
12. $-4x(3 - x^3)$
13. $(1 - x^3)(4x)$
14. $(-2x)(-3x)(5x + 2)$

In Exercises 15–24, perform the operations using the vertical format.

15. Add:
$$\begin{array}{r} 7x^3 - 2x^2 + 8 \\ \underline{-3x^3 \qquad\quad - 4} \end{array}$$

16. Add:
$$\begin{array}{r} 2x^5 - 3x^3 + 2x + 3 \\ \underline{4x^3 + \ x - 6} \end{array}$$

17. Subtract:
$$\begin{array}{r} 5x^2 - 3x + 8 \\ \underline{x - 3} \end{array}$$

18. Subtract:
$$\begin{array}{r} 0.6t^4 - \quad 2t^2 \qquad\quad \\ \underline{-t^4 + 0.5t^2 - 5.6} \end{array}$$

19. Multiply:
$$\begin{array}{r} -6x^2 + 15x - 4 \\ \underline{5x + 3} \end{array}$$

20. Multiply:
$$\begin{array}{r} 4x^4 + x^3 - 6x^2 + 9 \\ \underline{x^2 + 2x \ + 3} \end{array}$$

21. $(x^2 + 9)(x^2 - x - 4)$
22. $(x - 2)(x^2 + 2x + 4)$
23. $(x^2 - x + 1)(x^2 + x + 1)$
24. $(x^2 + 3x - 2)(x^2 - 3x - 2)$

In Exercises 25–52, find the product.

25. $(x + 3)(x + 4)$
26. $(x - 5)(x + 10)$
27. $(3x - 5)(2x + 1)$
28. $(7x - 2)(4x - 3)$
29. $(2x + 3)^2$
30. $(4x + 5)^2$
31. $(2x - 5y)^2$
32. $(5 - 8x)^2$
33. $[(x - 3) + y]^2$
34. $[(x + 1) - y]^2$
35. $(x + 10)(x - 10)$
36. $(2x + 3)(2x - 3)$
37. $(x + 2y)(x - 2y)$
38. $(2x + 3y)(2x - 3y)$
39. $[(m - 3) + n][(m - 3) - n]$
40. $[(x + y) + 1][(x + y) - 1]$
41. $(2r^2 - 5)(2r^2 + 5)$
42. $(3a^3 - 4b^2)(3a^3 + 4b^2)$
43. $(x + 1)^3$
44. $(x - 2)^3$
45. $(2x - y)^3$
46. $(3x + 2y)^3$
47. $(4x^3 - 3)^2$
48. $(8x + 3)^2$
49. $5x(x + 1) - 3x(x + 1)$
50. $(2x - 1)(x + 3) + 3(x + 3)$
51. $(u + 2)(u - 2)(u^2 + 4)$
52. $(x + y)(x - y)(x^2 + y^2)$

53. ***Think About It*** Must the sum of two second-degree polynomials be a second-degree polynomial? If not, give an example.

54. ***Think About It*** Is the product of two binomials always a binomial? Explain.

55. ***Find a Pattern*** Perform the multiplications.
   (a) $(x - 1)(x + 1)$
   (b) $(x - 1)(x^2 + x + 1)$
   (c) $(x - 1)(x^3 + x^2 + x + 1)$
   From the pattern formed by these products, can you predict the result of $(x - 1)(x^4 + x^3 + x^2 + x + 1)$?

56. ***Think About It*** When the polynomial $-x^3 + 3x^2 + 2x - 1$ is subtracted from an unknown polynomial, the difference is $5x^2 + 8$. If it is possible, find the unknown polynomial.

57. ***Compound Interest*** After 2 years, an investment of \$500 compounded annually at an interest rate r will yield an amount of
   $500(1 + r)^2$.
   (a) Write this polynomial in standard form.
   (b) Use a calculator to evaluate the expression for the values of r given in the table.

r	$5\frac{1}{2}\%$	7%	8%	$8\frac{1}{2}\%$	9%
$500(1 + r)^2$					

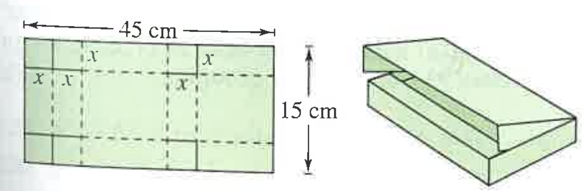
   (c) What conclusion can you make from the table?

58. ***Compound Interest*** After 3 years, an investment of \$1200 compounded annually at an interest rate r will yield an amount of
   $1200(1 + r)^3$.
   (a) Write this polynomial in standard form.
   (b) Use a calculator to evaluate the expression for the values of r given in the table.

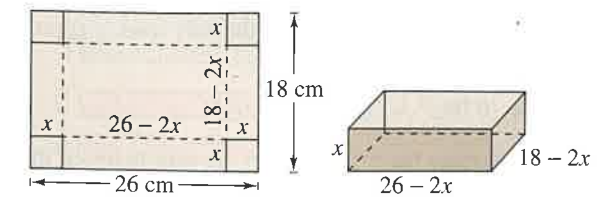
r	6%	7%	$7\frac{1}{2}\%$	8%	$8\frac{1}{2}\%$
$1200(1 + r)^3$					

   (c) What conclusion can you make from the table?

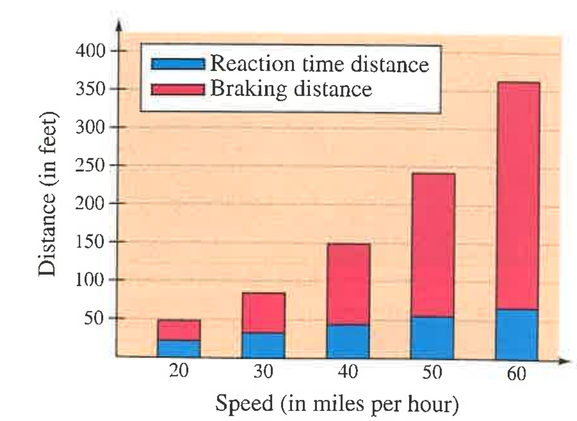
59. ***Volume of a Box*** A closed box is constructed by cutting along the solid lines and folding along the broken lines of the rectangular piece of metal shown in the figure. The length and width of the rectangle are 45 centimeters and 15 centimeters, respectively. Find the volume of the box in terms of x. Find the volume when $x = 3$, $x = 5$, and $x = 7$.



60. ***Volume of a Box*** An open box is made by cutting squares out of the corners of a piece of metal that is 18 centimeters by 26 centimeters (see figure). If the edge of each cut-out square is x inches, find the volume when $x = 1$, $x = 2$, and $x = 3$.



61. ***Stopping Distance*** The stopping distance of an automobile is the distance traveled during the driver's reaction time plus the distance traveled after the brakes are applied. In an experiment, these distances were measured (in feet) when the automobile was traveling at a speed of x miles per hour (see figure). The distance traveled during the reaction time was $R = 1.1x$, and the braking distance was $B = 0.14x^2 - 4.43x + 58.40$.
   (a) Determine the polynomial that represents the total stopping distance.
   (b) Use the result of part (a) to estimate the total stopping distance when $x = 30$, $x = 40$, and $x = 55$.
   (c) Use the bar graph to make a statement about the total stopping distance required for increasing speeds.



62. ***Safe Beam Load*** A uniformly distributed load is placed on a 1-inch-wide steel beam. When the span of the beam is x feet and its depth is 6 inches, the safe load is approximated by

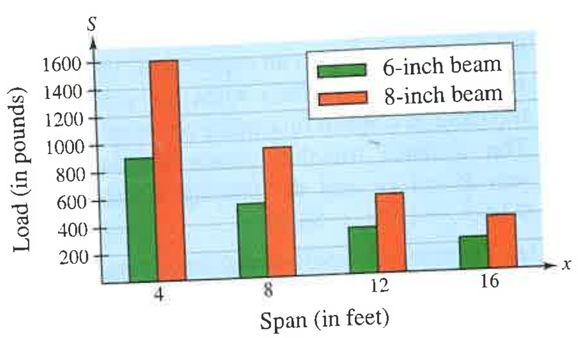
$S_6 = (0.06x^2 - 2.42x + 38.71)^2.$

When the depth is 8 inches, the safe load is approximated by

$S_8 = (0.08x^2 - 3.30x + 51.93)^2.$

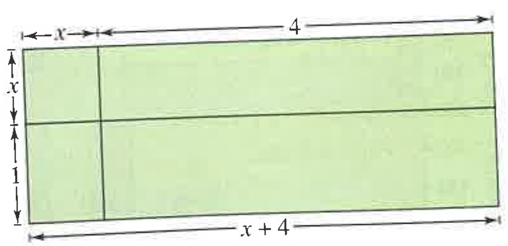
(a) Estimate the difference in the safe loads of these two beams when the span is 12 feet (see figure).

(b) How does the difference in safe load change as the span increases?

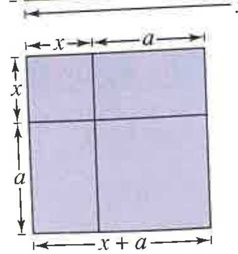


Geometrical Modeling **In Exercises 63 and 64, use the area model to write two different expressions for the area. Then equate the two expressions and name the algebraic property that is illustrated.**

63.



64.



In Exercises 65–68, factor out the common factor.

65. $3x + 6$

66. $5y - 30$

67. $2x^3 - 6x$

68. $4x^3 - 6x^2 + 12x$

In Exercises 69–74, factor the difference of two squares.

69. $x^2 - 36$

70. $x^2 - \frac{1}{4}$

71. $16y^2 - 9$

72. $49 - 9y^2$

73. $(x - 1)^2 - 4$

74. $25 - (z + 5)^2$

In Exercises 75–78, factor the perfect square trinomial.

75. $x^2 - 4x + 4$

76. $x^2 + 10x + 25$

77. $4t^2 + 4t + 1$

78. $9x^2 - 12x + 4$

In Exercises 79–88, factor the trinomial.

79. $x^2 + x - 2$

80. $x^2 + 5x + 6$

81. $s^2 - 5s + 6$

82. $t^2 - t - 6$

83. $20 - y - y^2$

84. $24 + 5z - z^2$

85. $3x^2 - 5x + 2$

86. $2x^2 - x - 1$

87. $5x^2 + 26x + 5$

88. $-5u^2 - 13u + 6$

In Exercises 89–92, factor the sum or difference of cubes.

89. $x^3 - 8$

90. $x^3 - 27$

91. $y^3 + 64$

92. $z^3 + 125$

In Exercises 93–96, factor by grouping.

93. $x^3 - x^2 + 2x - 2$

94. $x^3 + 5x^2 - 5x - 25$

95. $2x^3 - x^2 - 6x + 3$

96. $5x^3 - 10x^2 + 3x - 6$

In Exercises 97–128, completely factor the expression.

97. $x^3 - 9x$

98. $12x^2 - 48$

99. $x^3 - 4x^2$

100. $6x^2 - 54$

101. $x^2 - 2x + 1$

102. $16 + 6x - x^2$

103. $1 - 4x + 4x^2$

104. $-9x^2 + 6x - 1$

105. $2x^2 + 4x - 2x^3$

106. $2y^3 - 7y^2 - 15y$

107. $9x^2 + 10x + 1$

108. $13x + 6 + 5x^2$

109. $3x^3 + x^2 + 15x + 5$

110. $5 - x + 5x^2 - x^3$

111. $x^4 - 4x^3 + x^2 - 4x$

112. $3u - 2u^2 + 6 - u^3$

113. $25 - (z + 5)^2$

114. $(t - 1)^2 - 49$

115. $(x^2 + 1)^2 - 4x^2$

116. $(x^2 + 8)^2 - 36x^2$

117. $2t^3 - 16$

118. $5x^3 + 40$

119. $4x(2x - 1) + (2x - 1)^2$

120. $5(3 - 4x)^2 - 8(3 - 4x)(5x - 1)$

121. $2(x + 1)(x - 3)^2 - 3(x + 1)^2(x - 3)$

122. $7(3x + 2)^2(1 - x)^2 + (3x + 2)(1 - x)^3$

123. $7x(2)(x^2 + 1)(2x) - (x^2 + 1)^2(7)$

124. $3(x - 2)^2(x + 1)^4 + (x - 2)^3(4)(x + 1)^3$

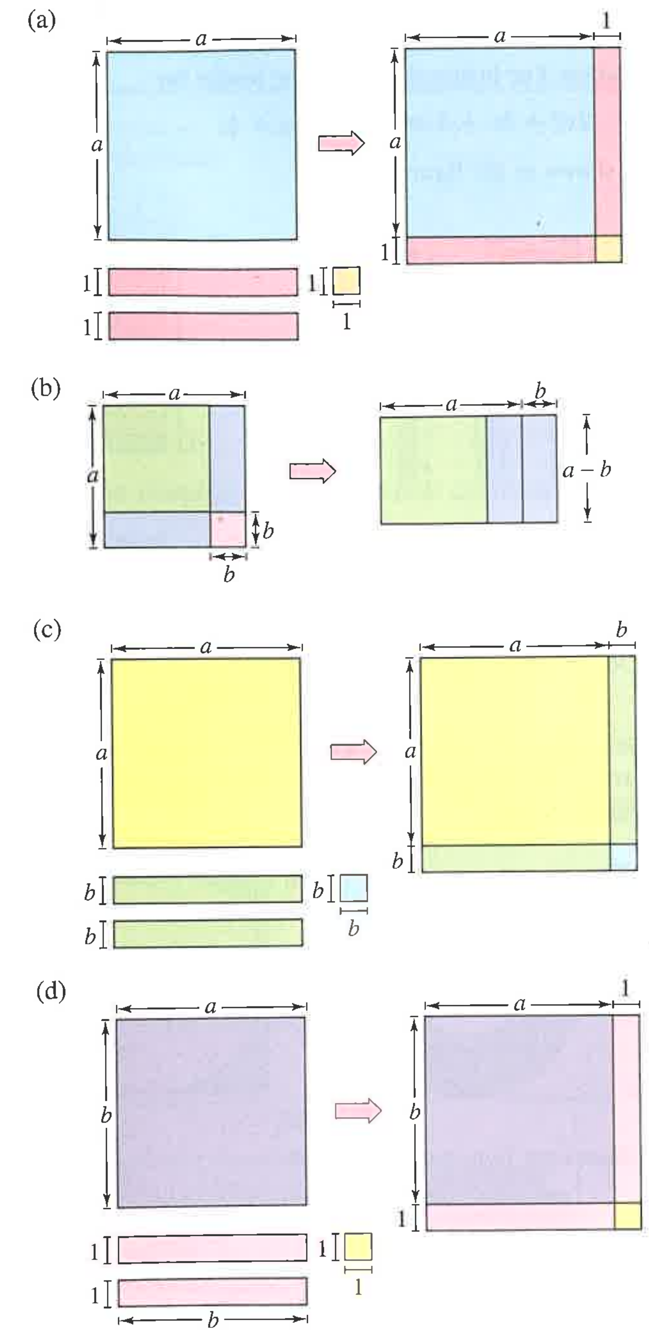
125. $2x(x - 5)^4 - x^2(4)(x - 5)^3$

126. $5(x^6 + 1)^4(6x^5)(3x + 2)^3 + 3(3x + 2)^2(3)(x^6 + 1)^5$

127. $\frac{x^2}{2}(x^2 + 1)^4 - (x^2 + 1)^5$

128. $5w^3(9w + 1)^4(9) + (2w + 1)^5(3w^2)$

Geometric Modeling **In Exercises 129–132, match the "geometric factoring model" with the correct factoring formula. [The models are labeled (a), (b), (c), and (d).]**



129. $a^2 - b^2 = (a + b)(a - b)$

130. $a^2 + 2ab + b^2 = (a + b)^2$

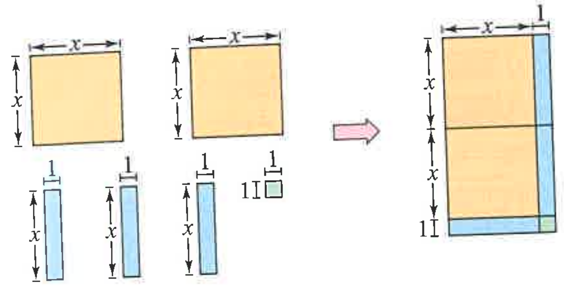
131. $a^2 + 2a + 1 = (a + 1)^2$

132. $ab + a + b + 1 = (a + 1)(b + 1)$

Geometric Modeling **In Exercises 133–136, draw a "geometric factoring model" to represent the factorization. For instance, a factoring model for**

$$2x^2 + 3x + 1 = (2x + 1)(x + 1)$$

is shown in the figure.



133. $3x^2 + 7x + 2 = (3x + 1)(x + 2)$

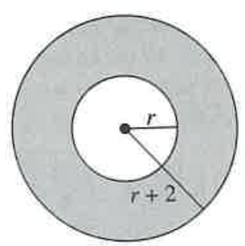
134. $x^2 + 4x + 3 = (x + 3)(x + 1)$

135. $2x^2 + 7x + 3 = (2x + 1)(x + 3)$

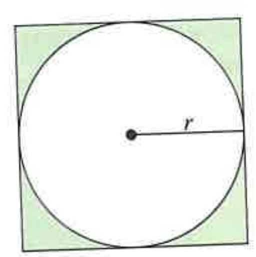
136. $x^2 + 3x + 2 = (x + 2)(x + 1)$

Geometry **In Exercises 137–140, write, in factored form, an expression for the shaded portion of the figure.**

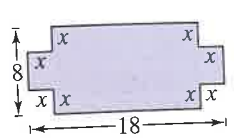
137.



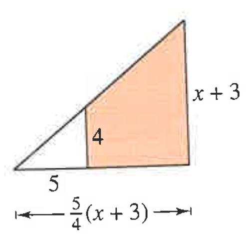
138.



139.



140.



In Exercises 141 and 142, find all values of b for which the trinomial can be factored.

141. $x^2 + bx - 15$

142. $x^2 + bx + 50$

In Exercises 143 and 144, find two integers c such that the trinomial can be factored. (There are many correct answers.)

143. $2x^2 + 5x + c$

144. $3x^2 - 10x + c$

145. ***Error Analysis*** Describe the error.

$$9x^2 - 9x - 54 = (3x + 6)(3x - 9)$$
$$= 3(x + 2)(x - 3)$$

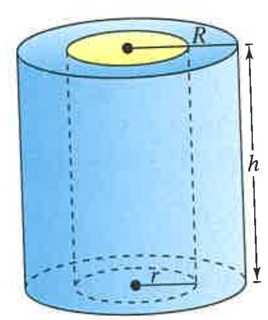
146. ***Think About It*** Is $(3x - 6)(x + 1)$ completely factored? Explain.

147. ***Geometry*** The cylindrical shell shown in the figure has a volume of

$$V = \pi R^2 h - \pi r^2 h.$$

(a) Factor the expression for the volume.

(b) From the result of part (a), show that the volume is 2π (average radius)(thickness of the shell)h.



148. ***Chemistry*** The rate of change of an autocatalytic chemical reaction is $kQx - kx^2$, where Q is the amount of the original substance, x is the amount of substance formed, and k is a constant of proportionality. Factor the expression.

P.4 Fractional Expressions

See Exercise 90 on page 51 for an example of how fractional expressions can be used to model the costs per ounce of precious metals from 1988 through 1992.

Domain of an Algebraic Expression ▫ *Simplifying Rational Expressions*
Operations with Rational Expressions ▫ *Compound Fractions*

Domain of an Algebraic Expression

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Two algebraic expressions are **equivalent** if they have the same domain and yield the same values for all numbers in their domain. For instance, $(x + 1) + (x + 2)$ and $2x + 3$ are equivalent.

EXAMPLE 1 *Finding the Domain of an Algebraic Expression*

a. The domain of the polynomial

$$2x^3 + 3x + 4$$

is the set of all real numbers. In fact, the domain of any polynomial is the set of all real numbers, *unless* the domain is specifically restricted.

b. The domain of the radical expression

$$\sqrt{x - 2}$$

is the set of real numbers greater than or equal to 2, because the square root of a negative number is not a real number.

c. The domain of the expression

$$\frac{x + 2}{x - 3}$$

is the set of all real numbers except $x = 3$, which would produce an undefined division by zero.

The quotient of two algebraic expressions is a **fractional expression.** Moreover, the quotient of two *polynomials* such as

$$\frac{1}{x}, \quad \frac{2x - 1}{x + 1}, \quad \text{or} \quad \frac{x^2 - 1}{x^2 + 1}$$

is a **rational expression.** Recall that a fraction is in reduced form if its numerator and denominator have no factors in common aside from ± 1. To write a fraction in reduced form, apply the following rule.

$$\frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b}, \qquad b \neq 0 \quad \text{and} \quad c \neq 0.$$

Study Tip

In Example 2, do not make the mistake of trying to reduce further by dividing *terms*.

$$\frac{x+6}{\cancel{3}} \neq \frac{x+\cancel{6}^{2}}{\cancel{3}} = x+2$$

Remember that to reduce fractions, divide common *factors*, not terms.

The key to success in simplifying rational expressions lies in your ability to *factor* polynomials.

EXAMPLE 2 ***Reducing a Rational Expression***

Write $\dfrac{x^2+4x-12}{3x-6}$ in reduced form.

Solution

$$\frac{x^2+4x-12}{3x-6} = \frac{(x+6)\cancel{(x-2)}}{3\cancel{(x-2)}} \qquad \text{Factor completely.}$$

$$= \frac{x+6}{3}, \quad x \neq 2 \qquad \text{Cancel common factors.}$$

Note that the original expression is undefined when $x = 2$ (because division by zero is undefined). To make sure that the reduced expression is *equivalent* to the original expression, you must restrict the domain of the reduced expression by excluding the value $x = 2$.

Simplifying Rational Expressions

When simplifying rational expressions, be sure to factor each polynomial completely before concluding that the numerator and denominator have no factors in common. Moreover, changing the sign of a factor may allow further reduction, as shown in part (b) of the next example.

EXAMPLE 3 ***Reducing Rational Expressions***

a. $$\frac{x^3-4x}{x^2+x-2} = \frac{x(x^2-4)}{(x+2)(x-1)}$$

$$= \frac{x\cancel{(x+2)}(x-2)}{\cancel{(x+2)}(x-1)} \qquad \text{Factor completely.}$$

$$= \frac{x(x-2)}{(x-1)}, \quad x \neq -2 \qquad \text{Cancel common factors.}$$

b. $$\frac{12+x-x^2}{2x^2-9x+4} = \frac{(4-x)(3+x)}{(2x-1)(x-4)} \qquad \text{Factor completely.}$$

$$= \frac{-\cancel{(x-4)}(3+x)}{(2x-1)\cancel{(x-4)}} \qquad (4-x) = -(x-4)$$

$$= -\frac{3+x}{2x-1}, \quad x \neq 4 \qquad \text{Cancel common factors.}$$

Operations with Rational Expressions

To multiply or divide rational expressions, we use the properties of fractions discussed in Section P.1. Recall that to divide fractions we invert the divisor and multiply.

EXAMPLE 4 ***Multiplying Rational Expressions***

$$\frac{2x^2+x-6}{x^2+4x-5}\cdot\frac{x^3-3x^2+2x}{4x^2-6x} = \frac{\cancel{(2x-3)}(x+2)}{(x+5)\cancel{(x-1)}}\cdot\frac{\cancel{x}(x-2)\cancel{(x-1)}}{2\cancel{x}\cancel{(2x-3)}}$$

$$= \frac{(x+2)(x-2)}{2(x+5)}, \quad x \neq 0, x \neq 1, x \neq \tfrac{3}{2}$$

EXAMPLE 5 ***Dividing Rational Expressions***

$$\frac{x^3-8}{x^2-4} \div \frac{x^2+2x+4}{x^3+8} = \frac{x^3-8}{x^2-4}\cdot\frac{x^3+8}{x^2+2x+4} \qquad \text{Invert and multiply.}$$

$$= \frac{\cancel{(x-2)}\cancel{(x^2+2x+4)}}{\cancel{(x+2)}\cancel{(x-2)}}\cdot\frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{x^2+2x+4}}$$

$$= x^2-2x+4, \quad x \neq \pm 2$$

To add or subtract rational expressions, you can use the LCD (least common denominator) method or the basic definition

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}, \quad b \neq 0 \text{ and } d \neq 0. \qquad \text{Basic definition}$$

This definition provides an efficient way of adding or subtracting *two* fractions that have no common factors in their denominators.

EXAMPLE 6 ***Subtracting Rational Expressions***

$$\frac{x}{x-3} - \frac{2}{3x+4} = \frac{x(3x+4)-2(x-3)}{(x-3)(3x+4)} \qquad \text{Basic definition}$$

$$= \frac{3x^2+4x-2x+6}{(x-3)(3x+4)} \qquad \text{Remove parentheses.}$$

$$= \frac{3x^2+2x+6}{(x-3)(3x+4)} \qquad \text{Combine like terms.}$$

For three or more fractions, or for fractions with a repeated factor in the denominators, the LCD method works well. Recall that the least common denominator of several fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator. Here is a numerical example.

$$\begin{aligned}\frac{1}{6}+\frac{3}{4}-\frac{2}{3} &= \frac{1\cdot 2}{6\cdot 2}+\frac{3\cdot 3}{4\cdot 3}-\frac{2\cdot 4}{3\cdot 4} &&\text{The LCD is 12.}\\ &= \frac{2}{12}+\frac{9}{12}-\frac{8}{12}\\ &= \frac{3}{12}\\ &= \frac{1}{4}\end{aligned}$$

NOTE Sometimes the numerator of the answer has a factor in common with the denominator. In such cases the answer should be reduced. For instance, in the example above, $\frac{3}{12}$ was reduced to $\frac{1}{4}$.

EXAMPLE 7 ***Combining Rational Expressions: The LCD Method***

Perform the operations and simplify.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{x^2-1}$$

Solution

Using the factored denominators $(x-1)$, x, and $(x+1)(x-1)$, you can see that the LCD is $x(x+1)(x-1)$.

$$\begin{aligned}&\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{(x+1)(x-1)}\\ &= \frac{3(x)(x+1)}{x(x+1)(x-1)}-\frac{2(x+1)(x-1)}{x(x+1)(x-1)}+\frac{(x+3)(x)}{x(x+1)(x-1)}\\ &= \frac{3(x)(x+1)-2(x+1)(x-1)+(x+3)(x)}{x(x+1)(x-1)}\\ &= \frac{3x^2+3x-2x^2+2+x^2+3x}{x(x+1)(x-1)}\\ &= \frac{2x^2+6x+2}{x(x+1)(x-1)}\\ &= \frac{2(x^2+3x+1)}{x(x+1)(x-1)}\end{aligned}$$

Compound Fractions

Fractional expressions with separate fractions in the numerator, denominator, or both, are called **compound** or **complex fractions.** Here are two examples.

$$\frac{\left(\frac{1}{x}\right)}{x^2+1} \quad\text{and}\quad \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2+1}\right)}$$

A compound fraction can be simplified by combining its numerator and denominator into single fractions, then inverting the denominator and multiplying.

EXAMPLE 8 ***Simplifying a Compound Fraction***

$$\begin{aligned}\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} &= \frac{\left[\frac{2-3(x)}{x}\right]}{\left[\frac{1(x-1)-1}{x-1}\right]} &&\text{Combine fractions.}\\ &= \frac{\left(\frac{2-3x}{x}\right)}{\left(\frac{x-2}{x-1}\right)} &&\text{Simplify.}\\ &= \frac{2-3x}{x}\cdot\frac{x-1}{x-2} &&\text{Invert and multiply.}\\ &= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x\neq 1\end{aligned}$$

Another way to simplify a compound fraction is to multiply each term in its numerator and denominator by the LCD of all fractions in its numerator and denominator. This method is applied to the fraction in Example 8 as follows.

$$\begin{aligned}\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} &= \frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)}\cdot\frac{x(x-1)}{x(x-1)}\\ &= \frac{2(x-1)-3x(x-1)}{x(x-1)-x}\\ &= \frac{-3x^2+5x-2}{x^2-2x}\\ &= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x\neq 1\end{aligned}$$

The next three examples illustrate some methods for simplifying fractional expressions involving radicals and negative exponents. These types of expressions occur frequently in calculus.

EXAMPLE 9 *Simplifying an Expression with Negative Exponents*

Simplify

$$x(1-2x)^{-3/2}+(1-2x)^{-1/2}.$$

Solution

By rewriting the expression with positive exponents, you obtain

$$\frac{x}{(1-2x)^{3/2}}+\frac{1}{(1-2x)^{1/2}}$$

which can then be combined by the LCD method. However, the process can be simplified by first removing the common factor with the *smaller exponent.*

$$\begin{aligned} x(1-2x)^{-3/2}+(1-2x)^{-1/2} &= (1-2x)^{-3/2}[x+(1-2x)^{(-1/2)-(-3/2)}] \\ &= (1-2x)^{-3/2}[x+(1-2x)^{1}] \\ &= \frac{1-x}{(1-2x)^{3/2}} \end{aligned}$$

NOTE In Example 9, note that when factoring, you subtract exponents

EXAMPLE 10 *Simplifying a Compound Fraction*

Simplify

$$\frac{(4-x^2)^{1/2}+x^2(4-x^2)^{-1/2}}{4-x^2}.$$

Solution

$$\begin{aligned} \frac{(4-x^2)^{1/2}+x^2(4-x^2)^{-1/2}}{4-x^2} &= \frac{(4-x^2)^{1/2}+x^2(4-x^2)^{-1/2}}{4-x^2}\cdot\frac{(4-x^2)^{1/2}}{(4-x^2)^{1/2}} \\ &= \frac{(4-x^2)^{1}+x^2(4-x^2)^{0}}{(4-x^2)^{3/2}} \\ &= \frac{4-x^2+x^2}{(4-x^2)^{3/2}} \\ &= \frac{4}{(4-x^2)^{3/2}} \end{aligned}$$

TECHNOLOGY

Some graphing utilities have a table feature that can be used to create tables of values. For instance, to evaluate the expression x^2-4 for $x=1, 2, 3, 4, 5, 6,$ and 7 on a *TI-83,* you can use the following keystrokes.

$\boxed{\text{Y=}}$ $\boxed{\text{CLEAR}}$
$\boxed{\text{X, T, }\theta\text{, n}}$ $\boxed{x^2}$ $\boxed{-}$ 4
$\boxed{\text{TBLSET}}$
TblStart=1
ΔTbl=1
Indpnt: Auto
Depend: Auto
$\boxed{\text{TABLE}}$

For the *TI-82,* use $\boxed{\text{X, T, }\theta}$ instead of $\boxed{\text{X, T, }\theta\text{, n}}$ and set TblMin = 1.

The table produced by these keystrokes is shown below.

X	Y1
1	-3
2	0
3	5
4	12
5	21
6	32
7	45

X=1

EXAMPLE 11 *Simplifying a Compound Fraction*

The expression from calculus

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

is an example of a *difference quotient.* Rewrite this expression by rationalizing its numerator.

Solution

$$\begin{aligned} \frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}, \quad h\neq 0 \end{aligned}$$

Notice that the original expression is meaningless when $h=0$, but the final expression *could* be evaluated when $h=0$.

GROUP ACTIVITY

COMPARING DOMAINS OF TWO EXPRESSIONS

Complete the following table by evaluating the expressions

$$\frac{x^2-3x+2}{x-2} \quad \text{and} \quad x-1$$

for the values of x. If you have a graphing utility with a *table feature,* use it to help create the table. Write a short paragraph describing the equivalence or nonequivalence of the two expressions.

x	-3	-2	-1	0	1	2	3
$\frac{x^2-3x+2}{x-2}$							
$x-1$							

WARM UP

Completely factor the polynomial.

1. $5x^2 - 15x^3$
2. $16x^2 - 9$
3. $9x^2 - 6x + 1$
4. $9 + 12y + 4y^2$
5. $z^2 + 4z + 3$
6. $x^2 - 15x + 50$
7. $3 + 8x - 3x^2$
8. $3x^2 - 46x + 15$
9. $s^3 + s^2 - 4s - 4$
10. $y^3 + 64$

P.4 Exercises

In Exercises 1–10, find the domain of the expression.

1. $3x^2 - 4x + 7$
2. $2x^2 + 5x - 2$
3. $4x^3 + 3, \quad x \geq 0$
4. $6x^2 - 9, \quad x > 0$
5. $\dfrac{1}{x-2}$
6. $\dfrac{x+1}{2x+1}$
7. $\dfrac{x-1}{x^2-4x}$
8. $\dfrac{2x+1}{x^2-9}$
9. $\sqrt{x+1}$
10. $\dfrac{1}{\sqrt{x+1}}$

In Exercises 11–16, find the missing factor in the numerator so that the two fractions will be equivalent.

11. $\dfrac{5}{2x} = \dfrac{5(\quad)}{6x^2}$
12. $\dfrac{3}{4} = \dfrac{3(\quad)}{4(x+1)}$
13. $\dfrac{x+1}{x} = \dfrac{(x+1)(\quad)}{x(x-2)}$
14. $\dfrac{3y-4}{y+1} = \dfrac{(3y-4)(\quad)}{y^2-1}$
15. $\dfrac{3x}{x-3} = \dfrac{3x(\quad)}{x^2-3x}$
16. $\dfrac{1-z}{z^2} = \dfrac{(1-z)(\quad)}{z^3+z^2}$

In Exercises 17–30, write the rational expression in reduced form.

17. $\dfrac{15x^2}{10x}$
18. $\dfrac{18y^2}{60y^5}$
19. $\dfrac{3xy}{xy+x}$
20. $\dfrac{9x^2+9x}{2x+2}$
21. $\dfrac{x-5}{10-2x}$
22. $\dfrac{x^2-25}{5-x}$
23. $\dfrac{x^3+5x^2+6x}{x^2-4}$
24. $\dfrac{x^2+8x-20}{x^2+11x+10}$
25. $\dfrac{y^2-7y+12}{y^2+3y-18}$
26. $\dfrac{3-x}{x^2+11x+10}$
27. $\dfrac{2-x+2x^2-x^3}{x-2}$
28. $\dfrac{x^2-9}{x^3+x^2-9x-9}$
29. $\dfrac{z^3-8}{z^2+2z+4}$
30. $\dfrac{y^3-2y^2-3y}{y^3+1}$

In Exercises 31 and 32, complete the table. What can you conclude?

31.

x	0	1	2	3	4	5	6
$\dfrac{x^2-2x-3}{x-3}$							
$x+1$							

32.

x	0	1	2	3	4	5	6
$\dfrac{x-3}{x^2-x-6}$							
$\dfrac{1}{x+2}$							

33. ***Error Analysis*** Describe the error.

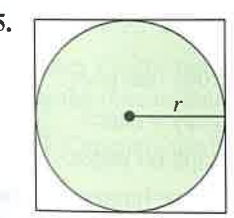
$$\frac{5x^3}{2x^3+4} = \frac{5x^3}{2x^3+4} = \frac{5}{2+4} = \frac{5}{6}$$

34. ***Think About It*** Is the following statement true for all nonzero real numbers a and b? Explain.

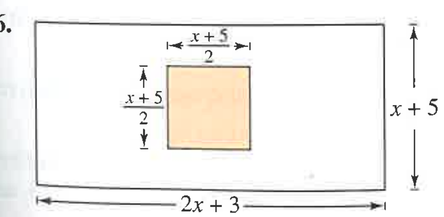
$$\frac{ax-b}{b-ax} = -1$$

In Exercises 35 and 36, find the ratio of the area of the shaded portion of the figure to the total area of the figure.

35.



36.



In Exercises 37–50, perform the multiplication or division and simplify.

37. $\dfrac{5}{x-1} \cdot \dfrac{x-1}{25(x-2)}$
38. $\dfrac{x+13}{x^3(3-x)} \cdot \dfrac{x(x-3)}{5}$
39. $\dfrac{(x+5)(x-3)}{x+2} \cdot \dfrac{1}{(x+5)(x+2)}$
40. $\dfrac{(x-9)(x+7)}{x+1} \cdot \dfrac{x}{9-x}$
41. $\dfrac{r}{r-1} \cdot \dfrac{r^2-1}{r^2}$
42. $\dfrac{4y-16}{5y+15} \cdot \dfrac{2y+6}{4-y}$
43. $\dfrac{t^2-t-6}{t^2+6t+9} \cdot \dfrac{t+3}{t^2-4}$
44. $\dfrac{y^3-8}{2y^3} \cdot \dfrac{4y}{y^2-5y+6}$
45. $\dfrac{x^2+xy-2y^2}{x^3+x^2y} \cdot \dfrac{x}{x^2+3xy+2y^2}$
46. $\dfrac{x^3-1}{x+1} \cdot \dfrac{x^2+1}{x^2-1}$
47. $\dfrac{3(x+y)}{4} \div \dfrac{x+y}{2}$
48. $\dfrac{x+2}{5(x-3)} \div \dfrac{x-2}{5(x-3)}$
49. $\dfrac{\left[\dfrac{x^2}{(x+1)^2}\right]}{\left[\dfrac{x}{(x+1)^3}\right]}$
50. $\dfrac{\left(\dfrac{x^2-1}{x}\right)}{\left[\dfrac{(x-1)^2}{x}\right]}$

In Exercises 51–64, perform the addition or subtraction and simplify.

51. $\dfrac{5}{x-1} + \dfrac{x}{x-1}$
52. $\dfrac{2x-1}{x+3} + \dfrac{1-x}{x+3}$
53. $6 - \dfrac{5}{x+3}$
54. $\dfrac{3}{x-1} - 5$
55. $\dfrac{3}{x-2} + \dfrac{5}{2-x}$
56. $\dfrac{2x}{x-5} - \dfrac{5}{5-x}$
57. $\dfrac{2}{x^2-4} - \dfrac{1}{x^2-3x+2}$
58. $\dfrac{x}{x^2+x-2} - \dfrac{1}{x+2}$

59. $\frac{1}{x^2 - x - 2} - \frac{x}{x^2 - 5x + 6}$

60. $\frac{2}{x^2 - x - 2} + \frac{10}{x^2 + 2x - 8}$

61. $-\frac{1}{x} + \frac{2}{x^2 + 1} + \frac{1}{x^3 + x}$

62. $\frac{2}{x + 1} + \frac{2}{x - 1} + \frac{1}{x^2 - 1}$

63. $x^2(x^2 + 1)^{-5} - (x^2 + 1)^{-4}$

64. $2x(x - 5)^{-3} - 4x^2(x - 5)^{-4}$

Error Analysis **In Exercises 65 and 66, describe the error.**

65. $$\frac{x + 4}{x + 2} - \frac{3x - 8}{x + 2} = \frac{x + 4 - 3x - 8}{x + 2} = \frac{-2x - 4}{x + 2} = \frac{-2(x + 2)}{x + 2} = -2$$

66. $$\frac{6 - x}{x(x + 2)} + \frac{x + 2}{x^2} + \frac{8}{x^2(x + 2)} = \frac{x(6 - x) + (x + 2)^2 + 8}{x^2(x + 2)} = \frac{6x - x^2 + x^2 + 4 + 8}{x^2(x + 2)} = \frac{6(x + 2)}{x^2(x + 2)} = \frac{6}{x^2}$$

In Exercises 67–80, simplify the compound fraction.

67. $\dfrac{\left(\frac{x}{2} - 1\right)}{(x - 2)}$

68. $\dfrac{(x - 4)}{\left(\frac{x}{4} - \frac{4}{x}\right)}$

69. $\dfrac{\left(\frac{1}{x} - \frac{1}{x + 1}\right)}{\left(\frac{1}{x + 1}\right)}$

70. $\dfrac{\left(\frac{5}{y} - \frac{6}{2y + 1}\right)}{\left(\frac{5}{y} + 4\right)}$

71. $\dfrac{\left(\frac{x + 3}{x - 3}\right)^2}{\frac{1}{x + 3} + \frac{1}{x - 3}}$

72. $\dfrac{\left(\frac{x + 4}{x + 5} - \frac{x}{x + 1}\right)}{4}$

73. $\dfrac{\left[\frac{1}{(x + h)^2} - \frac{1}{x^2}\right]}{h}$

74. $\dfrac{\left(\frac{x + h}{x + h + 1} - \frac{x}{x + 1}\right)}{h}$

75. $\dfrac{\left(\sqrt{x} - \frac{1}{2\sqrt{x}}\right)}{\sqrt{x}}$

76. $\dfrac{3x^{1/3} - x^{-2/3}}{3x^{-2/3}}$

77. $\dfrac{\left(\frac{t^2}{\sqrt{t^2 + 1}} - \sqrt{t^2 + 1}\right)}{t^2}$

78. $\dfrac{-x^3(1 - x^2)^{-1/2} - 2x(1 - x^2)^{1/2}}{x^4}$

79. $\dfrac{x(x + 1)^{-3/4} - (x + 1)^{1/4}}{x^2}$

80. $\dfrac{(2x + 1)^{1/3} - \frac{4x}{3(2x + 1)^{2/3}}}{(2x + 1)^{2/3}}$

In Exercises 81 and 82, rationalize the numerator of the expression.

81. $\frac{\sqrt{x + 2} - \sqrt{x}}{2}$

82. $\frac{\sqrt{z - 3} - \sqrt{z}}{3}$

83. ***Rate*** A photocopier copies at a rate of 16 pages per minute.

(a) Find the time required to copy one page.

(b) Find the time required to copy x pages.

(c) Find the time required to copy 60 pages.

84. ***Rate*** After working together for t hours on a common task, two workers have done fractional parts of the job equal to $t/3$ and $t/5$, respectively. What fractional part of the task has been completed?

85. ***Average*** Determine the average of the two real numbers $x/3$ and $2x/5$.

86. ***Partition into Equal Parts*** Find three real numbers that divide the real number line between $x/3$ and $3x/4$ into four equal parts.

Monthly Payment **In Exercises 87 and 88, use the formula that gives the approximate annual interest rate r of a monthly installment loan:**

$$r = \frac{\left[\frac{24(NM - P)}{N}\right]}{\left(P + \frac{NM}{12}\right)}$$

where N is the total number of payments, M is the monthly payment, and P is the amount financed.

87. (a) Approximate the annual interest rate for a 4-year car loan of \$15,000 that has monthly payments of \$400.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

88. (a) Approximate the annual interest rate for a 5-year car loan of \$18,000 that has monthly payments of \$400.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

89. ***Refrigeration*** When food (at room temperature) is placed in a refrigerator, the time required for the food to cool depends on the amount of food, the air circulation in the refrigerator, the original temperature of the food, and the temperature of the refrigerator. Consider the model that gives the temperature of food that is at 75°F and is placed in a 40°F refrigerator

$$T = 10\left(\frac{4t^2 + 16t + 75}{t^2 + 4t + 10}\right)$$

where T is the temperature in degrees Fahrenheit and t is the time in hours.

(a) Complete the table.

t	0	1	2	3	4	5
T						

(b) Create a bar graph showing the temperatures at the times given in the table in part (a).

90. ***Precious Metals*** The costs per fine ounce of gold and silver for the years 1988 through 1992 are given in the table. (Source: U.S. Bureau of Mines)

Year	1988	1989	1990	1991	1992
Gold	\$438	\$383	\$385	\$363	\$345
Silver	\$6.54	\$5.50	\$4.82	\$4.04	\$3.94

Mathematical models for this data are

$$\text{Cost of gold} = \frac{5301t + 37{,}498}{19t + 100}$$

and

$$\text{Cost of silver} = \frac{237t + 4734}{176t + 1000}$$

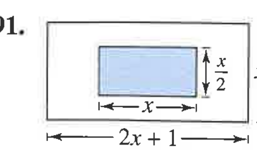
where $t = 0$ corresponds to the year 1990.

(a) Create a table using the models to estimate the prices of each metal for the given years. Compare the estimates given by the models with the actual prices.

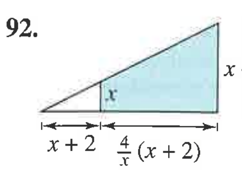
(b) Determine a model for the ratio of the price of gold to the price of silver. Use the model to find this ratio over the given years. Over this period of time, did the price of gold become more expensive or less expensive relative to the price of silver?

Probability **In Exercises 91 and 92, consider an experiment in which a marble is tossed into a box whose base is shown in the figure. The probability that the marble will come to rest in the shaded portion of the box is equal to the ratio of the shaded area to the total area of the figure. Find the probability.**

91.



92.



P.5 Solving Equations

See Exercise 42 on page 62 for an example of how a linear equation can be used to model the number of married women in the civilian work force in the United States

This ancient Egyptian papyrus discovered in 1858 contains one of the earliest examples of mathematical writing in existence. The papyrus itself dates back to around 1650 B.C., but it is actually a copy of writings from two centuries earlier. The algebraic equations on the papyrus were written in words. Diophantus, a Greek who lived around A.D. 250, is often called the Father of Algebra. He was the first to use abbreviated word forms in equations. *(Photo: © British Museum)*

Equations and Solutions of Equations

An **equation** is a statement that two algebraic expressions are equal. For example, $3x - 5 = 7$, $x^2 - x - 6 = 0$, and $\sqrt{2x} = 4$ are equations. To **solve** an equation in x means to find all values of x for which the equation is true. Such values are **solutions.** For instance, $x = 4$ is a solution of the equation $3x - 5 = 7$, because $3(4) - 5 = 7$ is a true statement.

The solutions of an equation depend on the kinds of numbers being considered. For instance, in the set of rational numbers, $x^2 = 10$ has no solution because there is no rational number whose square is 10. However, in the set of real numbers the equation has the two solutions $\sqrt{10}$ and $-\sqrt{10}$.

An equation that is true for *every* real number in the domain of the variable is called an **identity.** For example, $x^2 - 9 = (x + 3)(x - 3)$ is an identity because it is a true statement for any real value of x, and $x/(3x^2) = 1/(3x)$, where $x \neq 0$, is an identity because it is true for any nonzero real value of x.

An equation that is true for just *some* (or even none) of the real numbers in the domain of the variable is called a **conditional equation.** For example, the equation $x^2 - 9 = 0$ is conditional because $x = 3$ and $x = -3$ are the only values in the domain that satisfy the equation. Learning to solve conditional equations is the primary focus of this section.

Linear Equations

A **linear equation** in one variable x is an equation that can be written in the standard form

$$ax + b = 0$$

where a and b are real numbers, with $a \neq 0$. A linear equation has exactly one solution. To see this, consider the following steps. (Remember that $a \neq 0$.)

$ax + b = 0$	Original equation
$ax = -b$	Subtract b from both sides.
$x = -\dfrac{b}{a}$	Divide both sides by a.

To solve a conditional equation in x, isolate x on one side of the equation by a sequence of **equivalent** (and usually simpler) equations, each having the same solution(s) as the original equation. The operations that yield equivalent equations come from the properties of equality discussed in Section P.1.

Generating Equivalent Equations

An equation can be transformed into an *equivalent equation* by one or more of the following steps.

	Given Equation	Equivalent Equation
1. Remove symbols of grouping, combine like terms, or reduce fractions on one or both sides of the equation.	$2x - x = 4$	$x = 4$
2. Add (or subtract) the same quantity to (from) *both* sides of the equation.	$x + 1 = 6$	$x = 5$
3. Multiply (or divide) *both* sides of the equation by the same *nonzero* quantity.	$2x = 6$	$x = 3$
4. Interchange the two sides of the equation.	$2 = x$	$x = 2$

EXAMPLE 1 *Solving a Linear Equation*

Solve $3x - 6 = 0$.

Solution

$3x - 6 = 0$	Original equation
$3x = 6$	Add 6 to both sides.
$x = 2$	Divide both sides by 3.

Check: After solving an equation, you should **check each solution** in the *original* equation.

$3x - 6 = 0$	Original equation
$3(2) - 6 \stackrel{?}{=} 0$	Substitute 2 for x.
$0 = 0$	Solution checks. ✓

Study Tip

Students sometimes tell us that a solution looks easy when we work it out in class, but that they don't see where to begin when trying it alone. Keep in mind that no one—not even great mathematicians—can expect to look at every mathematical problem and immediately know where to begin. Many problems involve some trial and error before a solution is found. To make algebra work for you, you must put in a lot of time, you must expect to try solution methods that end up not working, and you must learn from both your successes and your failures.

To solve an equation involving fractional expressions, find the least common denominator of all terms and multiply every term by this LCD.

EXAMPLE 2 An Equation Involving Fractional Expressions

Solve $\frac{x}{3} + \frac{3x}{4} = 2$.

Solution

$$\frac{x}{3} + \frac{3x}{4} = 2 \qquad \text{Original equation}$$

$$(12)\frac{x}{3} + (12)\frac{3x}{4} = (12)2 \qquad \text{Multiply by the LCD of 12.}$$

$$4x + 9x = 24 \qquad \text{Reduce and multiply.}$$

$$13x = 24 \qquad \text{Combine like terms.}$$

$$x = \frac{24}{13} \qquad \text{Divide both sides by 13.}$$

The solution is $\frac{24}{13}$. Check this in the original equation.

When multiplying or dividing an equation by a *variable* quantity, it is possible to introduce an **extraneous** solution.

NOTE An extraneous solution is one that does not satisfy the original equation.

EXAMPLE 3 An Equation with an Extraneous Solution

Solve $\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$.

Solution

The LCD is $x^2 - 4$ or $(x+2)(x-2)$. Multiply every term by this LCD.

$$\frac{1}{x-2}(x+2)(x-2) = \frac{3}{x+2}(x+2)(x-2) - \frac{6x}{x^2-4}(x+2)(x-2)$$

$$x + 2 = 3(x-2) - 6x, \quad x \neq \pm 2$$

$$x + 2 = 3x - 6 - 6x$$

$$4x = -8$$

$$x = -2$$

In the original equation, $x = -2$ yields a denominator of zero. Therefore, $x = -2$ is an extraneous solution, and the original equation has *no solution.*

Quadratic Equations

A **quadratic equation** in x is an equation that can be written in the standard form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers, with $a \neq 0$. A quadratic equation in x is also known as a **second-degree polynomial equation in x.**

You should be familiar with the following four methods for solving quadratic equations.

SOLVING A QUADRATIC EQUATION

Method	*Example*
Factoring: If $ab = 0$, then $a = 0$ or $b = 0$.	$x^2 - x - 6 = 0$ $(x-3)(x+2) = 0$ $x - 3 = 0 \Rightarrow x = 3$ $x + 2 = 0 \Rightarrow x = -2$
Square Root Principle: If $u^2 = c$, where $c > 0$, then $u = \pm\sqrt{c}$.	$(x+3)^2 = 16$ $x + 3 = \pm 4$ $x = -3 \pm 4$ $x = 1$ or $x = -7$
Completing the Square: If $x^2 + bx = c$, then $x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$ $\left(x + \frac{b}{2}\right)^2 = c + \frac{b^2}{4}$.	$x^2 + 6x = 5$ $x^2 + 6x + 3^2 = 5 + 3^2$ $(x+3)^2 = 14$ $x + 3 = \pm\sqrt{14}$ $x = -3 \pm \sqrt{14}$
Quadratic Formula: If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.	$2x^2 + 3x - 1 = 0$ $x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-1)}}{2(2)}$ $= \frac{-3 \pm \sqrt{17}}{4}$

NOTE The Quadratic Formula can be derived by completing the square with the general form

$$ax^2 + bx + c = 0.$$

EXAMPLE 4 Solving Quadratic Equations by Factoring

a. $2x^2 + 9x + 7 = 3$ — Original equation

$2x^2 + 9x + 4 = 0$ — Standard form

$(2x + 1)(x + 4) = 0$ — Factored form

$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ — Set 1st factor equal to 0.

$x + 4 = 0 \Rightarrow x = -4$ — Set 2nd factor equal to 0.

The solutions are $-\frac{1}{2}$ and -4. Check these in the original equation.

b. $6x^2 - 3x = 0$ — Original equation

$3x(2x - 1) = 0$ — Factored form

$3x = 0 \Rightarrow x = 0$ — Set 1st factor equal to 0.

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ — Set 2nd factor equal to 0.

The solutions are 0 and $\frac{1}{2}$. Check these in the original equation.

Be sure you see that the Zero-Factor Property works *only* for equations written in standard form (in which the right side of the equation is zero). Therefore, all terms must be collected on one side *before* factoring. For instance, in the equation $(x - 5)(x + 2) = 8$ it is *incorrect* to set each factor equal to 8. Can you solve this equation correctly?

EXAMPLE 5 Extracting Square Roots

a. $4x^2 = 12$ — Original equation

$x^2 = 3$ — Divide both sides by 4.

$x = \pm\sqrt{3}$ — Extract square roots.

The solutions are $\sqrt{3}$ and $-\sqrt{3}$. Check these in the original equation.

b. $(x - 3)^2 = 7$ — Original equation

$x - 3 = \pm\sqrt{7}$ — Extract square roots.

$x = 3 \pm \sqrt{7}$ — Add 3 to both sides.

The solutions are $3 \pm \sqrt{7}$. Check these in the original equation.

EXAMPLE 6 The Quadratic Formula: Two Distinct Solutions

Use the Quadratic Formula to solve $x^2 + 3x = 9$.

Solution

$x^2 + 3x = 9$ — Original equation

$x^2 + 3x - 9 = 0$ — Standard form with $a = 1, b = 3, c = -9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-9)}}{2(1)}$$ Substitute.

$$x = \frac{-3 \pm \sqrt{45}}{2}$$ Simplify.

$$x = \frac{-3 \pm 3\sqrt{5}}{2}$$ Simplify.

The equation has two solutions:

$$x = \frac{-3 + 3\sqrt{5}}{2} \quad \text{and} \quad x = \frac{-3 - 3\sqrt{5}}{2}.$$

Check these in the original equation.

EXAMPLE 7 The Quadratic Formula: One Repeated Solution

Use the Quadratic Formula to solve $8x^2 - 24x + 18 = 0$.

Solution

$8x^2 - 24x + 18 = 0$ — Original equation

$4x^2 - 12x + 9 = 0$ — Standard form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$x = \frac{12 \pm \sqrt{144 - 4(4)(9)}}{2(4)}$$ Substitute.

$$x = \frac{12 \pm \sqrt{0}}{8}$$ Simplify.

$$x = \frac{3}{2}$$ Repeated solution

The solution is $\frac{3}{2}$. Check this in the original equation.

Polynomial Equations of Higher Degree

The methods used to solve quadratic equations can sometimes be extended to polynomials of higher degree.

EXAMPLE 8 ***Solving a Polynomial Equation by Factoring***

Solve $3x^4 = 48x^2$.

Solution

First write the polynomial equation in standard form with zero on one side, factor the other side, and then set each factor equal to zero.

$$\begin{aligned} 3x^4 &= 48x^2 && \text{Original equation} \\ 3x^4 - 48x^2 &= 0 && \text{Standard form} \\ 3x^2(x^2 - 16) &= 0 && \text{Factor.} \\ 3x^2(x + 4)(x - 4) &= 0 && \text{Factored form} \\ 3x^2 &= 0 \;\Rightarrow\; x = 0 && \text{Set 1st factor equal to 0.} \\ x + 4 &= 0 \;\Rightarrow\; x = -4 && \text{Set 2nd factor equal to 0.} \\ x - 4 &= 0 \;\Rightarrow\; x = 4 && \text{Set 3rd factor equal to 0.} \end{aligned}$$

Check:

$$\begin{aligned} 3x^4 &= 48x^2 && \text{Original equation} \\ 3(0)^4 &= 48(0)^2 && 0 \text{ checks. } \checkmark \\ 3(-4)^4 &= 48(-4)^2 && -4 \text{ checks. } \checkmark \\ 3(4)^4 &= 48(4)^2 && 4 \text{ checks. } \checkmark \end{aligned}$$

After checking, you can conclude that the solutions are 0, -4, and 4.

Study Tip

A common mistake that is made in solving an equation such as that in Example 8 is dividing both sides of the equation by the variable factor x^2. This loses the solution $x = 0$. When using factoring to solve an equation, be sure to set each factor equal to zero. Don't divide both sides of an equation by a variable factor in an attempt to simplify the equation.

EXAMPLE 9 ***Solving a Polynomial Equation by Factoring***

Solve $x^3 - 3x^2 - 3x + 9 = 0$.

Solution

$$\begin{aligned} x^3 - 3x^2 - 3x + 9 &= 0 && \text{Original equation} \\ x^2(x - 3) - 3(x - 3) &= 0 && \text{Factor by grouping.} \\ (x - 3)(x^2 - 3) &= 0 && \text{Distributive Property} \\ x - 3 &= 0 \;\Rightarrow\; x = 3 && \text{Set 1st factor equal to 0.} \\ x^2 - 3 &= 0 \;\Rightarrow\; x = \pm\sqrt{3} && \text{Set 2nd factor equal to 0.} \end{aligned}$$

The solutions are 3, $\sqrt{3}$, and $-\sqrt{3}$. Check these in the original equation.

Radical Equations

The steps involved in solving the remaining equations in this section will often introduce *extraneous solutions.* Operations such as squaring both sides of an equation, raising both sides of an equation to a rational power, or multiplying both sides by a variable quantity all have this potential danger. Thus, when you use any of these operations, checking is crucial.

EXAMPLE 10 ***Solving an Equation Involving a Rational Exponent***

Solve $4x^{3/2} - 8 = 0$.

Solution

$$\begin{aligned} 4x^{3/2} - 8 &= 0 && \text{Original equation} \\ 4x^{3/2} &= 8 && \text{Add 8 to both sides.} \\ x^{3/2} &= 2 && \text{Isolate } x^{3/2}. \\ x &= 2^{2/3} && \text{Raise both sides to } \tfrac{2}{3} \text{ power.} \\ x &\approx 1.587 && \text{Round to three decimal places.} \end{aligned}$$

The solution appears to be $2^{2/3}$. You can check this as follows.

Check:

$$\begin{aligned} 4x^{3/2} - 8 &= 0 && \text{Original equation} \\ 4(2^{2/3})^{3/2} &\stackrel{?}{=} 8 && \text{Substitute } 2^{2/3} \text{ for } x. \\ 4(2) &\stackrel{?}{=} 8 && \text{Property of exponents} \\ 8 &= 8 && \text{Solution checks. } \checkmark \end{aligned}$$

NOTE The essential technique used in Example 10 is to isolate the factor with the rational exponent, and raise both sides to the *reciprocal power.* In Example 11, this is equivalent to isolating the square root and squaring both sides.

EXAMPLE 11 ***Solving an Equation Involving a Radical***

$$\begin{aligned} \sqrt{2x + 7} - x &= 2 && \text{Original equation} \\ \sqrt{2x + 7} &= x + 2 && \text{Isolate the square root.} \\ 2x + 7 &= x^2 + 4x + 4 && \text{Square both sides.} \\ 0 &= x^2 + 2x - 3 && \text{Standard form} \\ 0 &= (x + 3)(x - 1) && \text{Factored form} \\ x + 3 &= 0 \;\Rightarrow\; x = -3 && \text{Set 1st factor equal to 0.} \\ x - 1 &= 0 \;\Rightarrow\; x = 1 && \text{Set 2nd factor equal to 0.} \end{aligned}$$

By checking these values, you can determine that the only solution is 1.

Absolute Value Equations

To solve an equation involving an absolute value, remember that the expression inside the absolute value signs can be positive or negative. This results in *two* separate equations, each of which must be solved.

EXAMPLE 12 *Solving an Equation Involving Absolute Value*

Solve $|x^2 - 3x| = -4x + 6$.

Solution

First Equation

$x^2 - 3x = -4x + 6$ — Use positive expression.

$x^2 + x - 6 = 0$ — Standard form

$(x + 3)(x - 2) = 0$ — Factored form

$x + 3 = 0 \Rightarrow x = -3$ — Set 1st factor equal to 0.

$x - 2 = 0 \Rightarrow x = 2$ — Set 2nd factor equal to 0.

Second Equation

$-(x^2 - 3x) = -4x + 6$ — Use negative expression.

$x^2 - 7x + 6 = 0$ — Standard form

$(x - 1)(x - 6) = 0$ — Factored form

$x - 1 = 0 \Rightarrow x = 1$ — Set 1st factor equal to 0.

$x - 6 = 0 \Rightarrow x = 6$ — Set 2nd factor equal to 0.

Of the possible solutions -3, 2, 1, and 6, a check will show that only -3 and 1 are actual solutions.

GROUP ACTIVITY
SOLVING EQUATIONS

Choose one of the equations below and write a step-by-step explanation of how to solve the equation, without using another equation in the explanation. Exchange explanations with another student— see if he or she can correctly solve the equation just by following your instructions.

a. $x - 2 + \dfrac{3x - 1}{8} = \dfrac{x + 4}{4}$ **b.** $t - \{7 - [t - (7 + t)]\} = 27$

WARM UP

Perform the operations and simplify your answer.

1. $(2x - 4) - (5x + 6)$
2. $(3x - 5) + (2x - 7)$
3. $2(x + 1) - (x + 2)$
4. $-3(2x - 4) + 7(x + 2)$
5. $\dfrac{x}{3} + \dfrac{x}{5}$
6. $x - \dfrac{x}{4}$
7. $\dfrac{1}{x + 1} - \dfrac{1}{x}$
8. $\dfrac{2}{x} + \dfrac{3}{x}$
9. $\dfrac{4}{x} + \dfrac{3}{x - 2}$
10. $\dfrac{1}{x + 1} - \dfrac{1}{x - 1}$

P.5 Exercises

In Exercises 1–6, determine whether the values of x are solutions of the equation.

Equation	*Values*	
1. $5x - 3 = 3x + 5$	(a) $x = 0$	(b) $x = -5$
	(c) $x = 4$	(d) $x = 10$
2. $7 - 3x = 5x - 17$	(a) $x = -3$	(b) $x = 0$
	(c) $x = 8$	(d) $x = 3$
3. $3x^2 + 2x - 5 = 2x^2 - 2$	(a) $x = -3$	(b) $x = 1$
	(c) $x = 4$	(d) $x = -5$
4. $5x^3 + 2x - 3 = 4x^3 + 2x - 11$	(a) $x = 2$	(b) $x = -2$
	(c) $x = 0$	(d) $x = 10$
5. $\dfrac{5}{2x} - \dfrac{4}{x} = 3$	(a) $x = -\frac{1}{2}$	(b) $x = 4$
	(c) $x = 0$	(d) $x = \frac{1}{4}$
6. $\sqrt[3]{x - 8} = 3$	(a) $x = 2$	(b) $x = -5$
	(c) $x = 35$	(d) $x = 8$

In Exercises 7–12, determine whether the equation is an identity or a conditional equation.

7. $2(x - 1) = 2x - 2$
8. $3(x + 2) = 5x + 4$
9. $-6(x - 3) + 5 = -2x + 10$
10. $3(x + 2) - 5 = 3x + 1$
11. $x^2 - 8x + 5 = (x - 4)^2 - 11$
12. $3 + \dfrac{1}{x + 1} = \dfrac{4x}{x + 1}$

13. ***Think About It***
   (a) What is meant by equivalent equations? Give an example of two equivalent equations.
   (b) In your own words, describe the steps used to transform an equation into an equivalent equation.

14. Justify each step of the solution.

$$3(x - 4) + 10 = 7$$
$$3x - 12 + 10 = 7$$
$$3x - 2 = 7$$
$$3x - 2 + 2 = 7 + 2$$
$$3x = 9$$
$$\frac{3x}{3} = \frac{9}{3}$$
$$x = 3$$

In Exercises 15–40, solve the equation (if possible) and check your solution.

15. $2(x + 5) - 7 = 3(x - 2)$

16. $2(13t - 15) + 3(t - 19) = 0$

17. $\frac{5x}{4} + \frac{1}{2} = x - \frac{1}{2}$

18. $\frac{x}{5} - \frac{x}{2} = 3$

19. $0.25x + 0.75(10 - x) = 3$

20. $0.60x + 0.40(100 - x) = 50$

21. $x + 8 = 2(x - 2) - x$

22. $3(x + 3) = 5(1 - x) - 1$

23. $\frac{100 - 4u}{3} = \frac{5u + 6}{4} + 6$

24. $\frac{17 + y}{y} + \frac{32 + y}{y} = 100$

25. $\frac{5x - 4}{5x + 4} = \frac{2}{3}$

26. $\frac{10x + 3}{5x + 6} = \frac{1}{2}$

27. $10 - \frac{13}{x} = 4 + \frac{5}{x}$

28. $\frac{15}{x} - 4 = \frac{6}{x} + 3$

29. $\frac{1}{x - 3} + \frac{1}{x + 3} = \frac{10}{x^2 - 9}$

30. $\frac{1}{x - 2} + \frac{3}{x + 3} = \frac{4}{x^2 + x - 6}$

31. $\frac{x}{x + 4} + \frac{4}{x + 4} + 2 = 0$

32. $\frac{2}{(x - 4)(x - 2)} = \frac{1}{x - 4} + \frac{2}{x - 2}$

33. $\frac{7}{2x + 1} - \frac{8x}{2x - 1} = -4$

34. $\frac{4}{u - 1} + \frac{6}{3u + 1} = \frac{15}{3u + 1}$

35. $(x + 2)^2 + 5 = (x + 3)^2$

36. $(x + 1)^2 + 2(x - 2) = (x + 1)(x - 2)$

37. $(x + 2)^2 - x^2 = 4(x + 1)$

38. $(2x + 1)^2 = 4(x^2 + x + 1)$

39. $4 - 2(x - 2b) = ax + 3$

40. $5 + ax = 12 - bx$

41. ***Exploration***

(a) Complete the table.

x	-1	0	1	2	3	4
$3.2x - 5.8$						

(b) Use the table in part (a) to determine the interval in which the solution to the equation $3.2x - 5.8 = 0$ is located. Explain your reasoning.

(c) Complete the table.

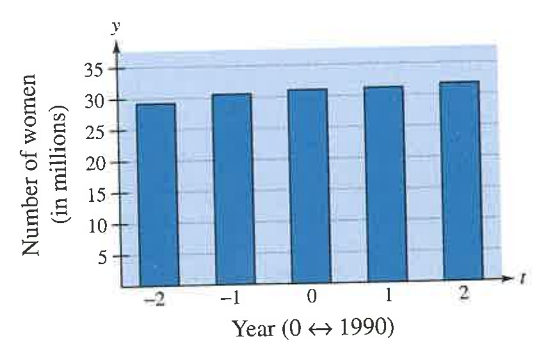
x	1.5	1.6	1.7	1.8	1.9	2
$3.2x - 5.8$						

(d) Use the table in part (c) to determine the interval in which the solution to the equation $3.2x - 5.8 = 0$ is located. Explain how this process can be used to approximate the solution to any desired degree of accuracy.

42. ***Using a Model*** The number of married women y in the civilian work force (in millions) in the United States from 1988 to 1992 can be approximated by the model

$y = 0.43t + 30.86$

where $t = 0$ represents 1990 (see figure). According to this model, during which year did this number reach 30 million? Explain how to answer the question graphically and algebraically. (Source: U.S. Bureau of Labor Statistics)



In Exercises 43–56, solve the equation by factoring.

43. $6x^2 + 3x = 0$

44. $9x^2 - 1 = 0$

45. $x^2 - 2x - 8 = 0$

46. $x^2 + 10x + 25 = 0$

47. $3 + 5x - 2x^2 = 0$

48. $16x^2 + 56x + 49 = 0$

49. $2x^2 = 19x + 33$

50. $(x + a)^2 - b^2 = 0$

51. $2x^4 - 18x^2 = 0$

52. $20x^3 - 125x = 0$

53. $x^3 - 2x^2 - 3x = 0$

54. $x^3 - 3x^2 - x + 3 = 0$

55. $2x^4 - 15x^3 + 18x^2 = 0$

56. $x^3 + 2x^2 - 3x - 6 = 0$

In Exercises 57–64, solve the equation by extracting square roots. List both the exact solution *and* the decimal solution rounded to two decimal places.

57. $x^2 = 16$

58. $x^2 = 144$

59. $3x^2 = 36$

60. $9x^2 = 25$

61. $(x - 12)^2 = 18$

62. $(x + 13)^2 = 21$

63. $(x + 2)^2 = 12$

64. $(x - 5)^2 = 20$

In Exercises 65–70, solve the quadratic equation by completing the square.

65. $x^2 - 2x = 0$

66. $x^2 + 4x = 0$

67. $x^2 + 6x + 2 = 0$

68. $x^2 + 8x + 14 = 0$

69. $8 + 4x - x^2 = 0$

70. $4x^2 - 4x - 99 = 0$

In Exercises 71–82, use the Quadratic Formula to solve the equation.

71. $2x^2 + x - 1 = 0$

72. $2x^2 - x - 1 = 0$

73. $x^2 + 8x - 4 = 0$

74. $4x^2 - 4x - 4 = 0$

75. $12x - 9x^2 = -3$

76. $16x^2 + 22 = 40x$

77. $3x + x^2 - 1 = 0$

78. $36x^2 + 24x - 7 = 0$

79. $28x - 49x^2 = 4$

80. $9x^2 + 24x + 16 = 0$

81. $8t = 5 + 2t^2$

82. $25h^2 + 80h + 61 = 0$

83. ***True or False?*** If $(2x - 3)(x + 5) = 8$, then $2x - 3 = 8$ or $x + 5 = 8$. Explain.

84. ***Exploration*** Solve the equation

$3(x + 4)^2 + (x + 4) - 2 = 0$

in two ways.

(a) Let $u = x + 4$, and solve the resulting equation for u. Then solve the u-solution for x.

(b) Expand and collect like terms in the equation, and solve the resulting equation for x.

(c) Which method is easier? Explain.

85. ***Exploration*** Solve the equations, given that a and b are not zero.

(a) $ax^2 + bx = 0$

(b) $ax^2 - ax = 0$

86. ***Dimensions of a Building*** The floor of a one-story building is 14 feet longer than it is wide. The building has 1632 square feet of floor space.

(a) Draw a rectangle that gives a visual representation of the floor space. Represent the width as w and show the length in terms of w.

(b) Write a quadratic equation in terms of w.

(c) Find the length and width of the building floor.

In Exercises 87–94, solve the equation of quadratic type. Check your solutions in the original equation.

87. $x^4 - 4x^2 + 3 = 0$

88. $4x^4 - 65x^2 + 16 = 0$

89. $\frac{1}{t^2} + \frac{8}{t} + 15 = 0$

90. $6\left(\frac{s}{s + 1}\right)^2 + 5\left(\frac{s}{s + 1}\right) - 6 = 0$

91. $2x + 9\sqrt{x} = 5$

92. $6x - 7\sqrt{x} - 3 = 0$

93. $3x^{1/3} + 2x^{2/3} = 5$

94. $9t^{2/3} + 24t^{1/3} + 16 = 0$

In Exercises 95–108, find all solutions of the equation. Check your solutions in the original equation.

95. $\sqrt{x-10} - 4 = 0$ **96.** $\sqrt{5-x} - 3 = 0$

97. $\sqrt[3]{2x+5} + 3 = 0$ **98.** $\sqrt[3]{3x+1} - 5 = 0$

99. $x = \sqrt{11x - 30}$ **100.** $2x - \sqrt{15 - 4x} = 0$

101. $\sqrt{x+1} - 3x = 1$ **102.** $\sqrt{x+5} = \sqrt{x-5}$

103. $\sqrt{x} - \sqrt{x-5} = 1$ **104.** $\sqrt{x} + \sqrt{x-20} = 10$

105. $2\sqrt{x+1} - \sqrt{2x+3} = 1$

106. $3\sqrt{x} - \dfrac{4}{\sqrt{x}} = 4$

107. $(x-5)^{2/3} = 16$

108. $(x+3)^{3/4} = 27$

109. ***Market Research*** The demand equation for a certain product is modeled by $p = 40 - \sqrt{0.01x + 1}$, where x is the number of units demanded per day and p is the price per unit. Approximate the demand if the price is \$37.55.

110. ***Market Research*** The demand equation for a certain product is modeled by $p = 40 - \sqrt{0.0001x + 1}$, where x is the number of units demanded per day and p is the price per unit. Approximate the demand if the price is \$34.70.

In Exercises 111 and 112, solve for the indicated variable.

111. ***Surface Area of a Cone***

Solve for h: $S = \pi r\sqrt{r^2 + h^2}$

112. ***Inductance***

Solve for Q: $i = \pm\sqrt{\dfrac{1}{LC}}\sqrt{Q^2 - q}$

In Exercises 113 and 114, consider an equation of the form $x + \sqrt{x - a} = b$, where a and b are constants.

113. ***Exploration*** Find a and b if the solution to the equation is $x = 20$. (There are many correct answers.)

114. ***Essay*** Write a short paragraph listing the steps required in solving an equation involving radicals.

In Exercises 115–120, find all solutions of the equation. Check your solutions in the original equation.

115. $|x+1| = 2$ **116.** $|x-2| = 3$

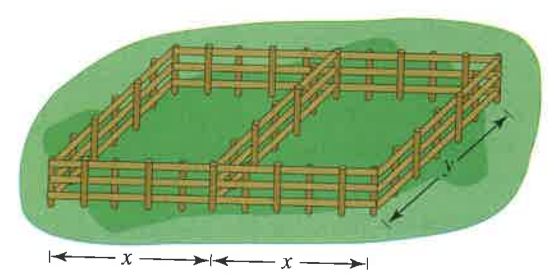
117. $|2x-1| = 5$ **118.** $|3x+2| = 7$

119. $|x^2 + 6x| = 3x + 18$ **120.** $|x - 10| = x^2 - 10x$

Think About It **In Exercises 121 and 122, find an equation having the given solutions. (There are many correct answers.)**

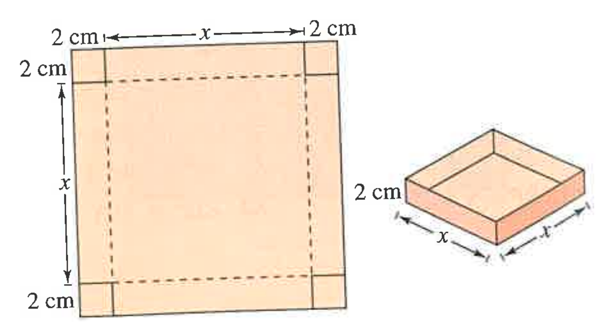
121. $-3, 5$ **122.** $0, 2, \frac{5}{2}$

123. ***Dimensions of a Corral*** A rancher has 100 meters of fencing to enclose two adjacent rectangular corrals (see figure). Find the dimensions such that the enclosed area will be 350 square meters.



$4x + 3y = 100$

124. ***Dimensions of a Box*** An open box is to be made from a square piece of material by cutting 2-centimeter squares from the corners and turning up the sides (see figure). The volume of the finished box is to be 200 cubic centimeters. Find the size of the original piece of material.



P.6 Solving Inequalities

Introduction ▫ *Properties of Inequalities* ▫ *Linear Inequalities* ▫ *Absolute Value Inequalities* ▫ *Other Types of Inequalities*

See Exercise 97 on page 76 for an example of how a quadratic inequality can be used to model the percent of the American population that are college graduates.

Introduction

Simple inequalities were reviewed in Section P.1. There, you used inequality symbols $<, \leq, >$, and $\geq$ to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality

$$x \geq 3$$

denotes all real numbers x that are greater than or equal to 3.

In this section you will expand your work with inequalities to include more involved statements such as

$$5x - 7 < 3x + 9 \quad \text{and} \quad -3 \leq 6x - 1 < 3.$$

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. Such values are **solutions** and are said to **satisfy** the inequality. The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality. For instance, the solution set of $x + 1 < 4$ is all real numbers that are less than 3.

The set of all points on the real number line that represent the solution set is the **graph** of the inequality. Graphs of many types of inequalities consist of intervals on the real number line. You can review the nine basic types of intervals on the real number line by turning to pages 3 and 4 in Section P.1. On those pages, note that each type of interval can be classified as *bounded* or *unbounded.*

EXAMPLE 1 Intervals and Inequalities

Write an inequality to represent each interval and state whether the interval is bounded or unbounded.

a. $(-3, 5]$ **b.** $(-3, \infty)$ **c.** $[0, 2]$

Solution

a. $(-3, 5]$ corresponds to $-3 < x \leq 5$. Bounded

b. $(-3, \infty)$ corresponds to $-3 < x$. Unbounded

c. $[0, 2]$ corresponds to $0 \leq x \leq 2$. Bounded

Properties of Inequalities

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable you can make use of the **properties of inequalities.** These properties are similar to the properties of equality, but there are two important exceptions. When both sides of an inequality are multiplied or divided by a negative number, the direction of the inequality symbol must be reversed. Here is an example.

$-2 < 5$	Original inequality
$(-3)(-2) > (-3)(5)$	Multiply both sides by -3 and reverse inequality.
$6 > -15$	

Two inequalities that have the same solution set are **equivalent.** For instance, the inequalities

$$x + 2 < 5 \quad \text{and} \quad x < 3$$

are equivalent. To obtain the second inequality from the first, you can subtract 2 from each side of the inequality. The following list describes the operations that can be used to create equivalent inequalities.

PROPERTIES OF INEQUALITIES

Let a, b, c, and d be real numbers.

1. *Transitive Property*

   $a < b$ and $b < c \Rightarrow a < c$

2. *Addition of Inequalities*

   $a < b$ and $c < d \Rightarrow a + c < b + d$

3. *Addition of a Constant*

   $a < b \Rightarrow a + c < b + c$

4. *Multiplication by a Constant*

   For $c > 0$, $a < b \Rightarrow ac < bc$

   For $c < 0$, $a < b \Rightarrow ac > bc$

NOTE Each of the properties above is true if the symbol $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$. For instance, another form of the multiplication property would be as follows.

For $c > 0$, $a \leq b \Rightarrow ac \leq bc$

For $c < 0$, $a \leq b \Rightarrow ac \geq bc$

Linear Inequalities

The simplest type of inequality is a **linear inequality** in a single variable. For instance, $2x + 3 > 4$ is a linear inequality in x.

In the following examples, pay special attention to the steps in which the inequality symbol is reversed. Remember that when you multiply or divide by a negative number, you must reverse the inequality symbol.

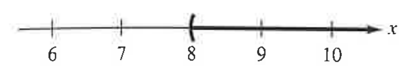
EXAMPLE 2 *Solving a Linear Inequality*

Solve $5x - 7 > 3x + 9$.

Solution

$5x - 7 > 3x + 9$	Original inequality
$5x > 3x + 16$	Add 7 to both sides.
$5x - 3x > 16$	Subtract $3x$ from both sides.
$2x > 16$	Combine like terms.
$x > 8$	Divide both sides by 2.

The solution set is all real numbers that are greater than 8, which is denoted by $(8, \infty)$. The graph is shown in Figure P.6.



Solution interval: $(8, \infty)$

FIGURE P.6

Checking the solution set of an inequality is not as simple as checking the solutions of an equation. You can, however, get an indication of the validity of a solution set by substituting a few convenient values of x.

EXAMPLE 3 *Solving a Linear Inequality*

Solve $1 - \dfrac{3x}{2} \geq x - 4$.

Solution

$1 - \dfrac{3x}{2} \geq x - 4$	Original inequality
$2 - 3x \geq 2x - 8$	Multiply both sides by 2.
$-3x \geq 2x - 10$	Subtract 2 from both sides.
$-5x \geq -10$	Subtract $2x$ from both sides.
$x \leq 2$	Divide both sides by -5 and reverse inequality.

The solution set is all real numbers that are less than or equal to 2, which is denoted by $(-\infty, 2]$. The graph is shown in Figure P.7.

Solution interval: $(-\infty, 2]$

FIGURE P.7

Sometimes it is possible to write two inequalities as a **double inequality.** For instance, you can write the two inequalities $-4 \le 5x - 2$ and $5x - 2 < 7$ more simply as

$$-4 \le 5x - 2 < 7.$$

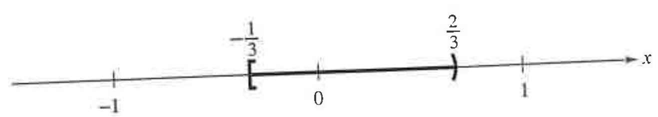
This form allows you to solve the two inequalities together, as demonstrated in Example 4.

EXAMPLE 4 *Solving a Double Inequality*

To solve the double inequality, you can isolate x as the middle term.

$-3 \le 6x - 1 < 3$	Original inequality
$-2 \le 6x < 4$	Add 1 to all three parts.
$-\frac{1}{3} \le x < \frac{2}{3}$	Divide all three parts by 6 and reduce.

The solution set is all real numbers that are greater than or equal to $-\frac{1}{3}$ and less than $\frac{2}{3}$, which is denoted by $\left[-\frac{1}{3}, \frac{2}{3}\right)$. The graph is shown in Figure P.8.



Solution interval: $\left[-\frac{1}{3}, \frac{2}{3}\right)$

FIGURE P.8

The double inequality in Example 4 could have been solved in two parts as follows.

$-3 \le 6x - 1$	and	$6x - 1 < 3$
$-2 \le 6x$		$6x < 4$
$-\frac{1}{3} \le x$		$x < \frac{2}{3}$

The solution set consists of all real numbers that satisfy *both* inequalities. In other words, the solution set is the set of all values of x for which $-\frac{1}{3} \le x < \frac{2}{3}$.

When combining two inequalities to form a double inequality, be sure that the inequalities satisfy the Transitive Property. For instance, it is *incorrect* to combine the inequalities $3 < x$ and $x \le -1$ as $3 < x \le -1$. This "inequality" is obviously wrong, because 3 is not less than -1.

Absolute Value Inequalities

SOLVING AN ABSOLUTE VALUE INEQUALITY

Let x be a variable of an algebraic expression and let a be a real number such that $a \ge 0$.

1. The solutions of $|x| < a$ are all values of x that lie between $-a$ and a.

$$|x| < a \quad \text{if and only if} \quad -a < x < a.$$

2. The solutions of $|x| > a$ are all values of x that are less than $-a$ or greater than a.

$$|x| > a \quad \text{if and only if} \quad x < -a \quad \text{or} \quad x > a.$$

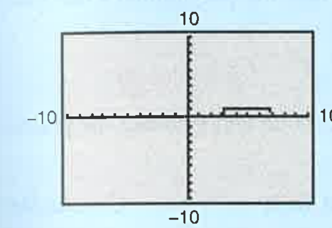
These rules are also valid if $<$ is replaced by $\le$ and $>$ is replaced by $\ge$.

TECHNOLOGY

A graphing utility can be used to give a rough indication of the graph of an inequality. For instance, on a *TI-83* or a *TI-82*, you can graph $|x - 5| < 2$ (see Example 5) by entering

Y1 = abs (X − 5) < 2

and pressing the graph key. With a standard setting, the graph should look like that shown below.



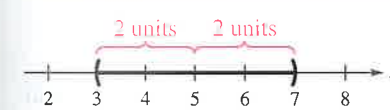
EXAMPLE 5 *Solving an Absolute Value Inequality*

Solve $|x - 5| < 2$.

Solution

$|x - 5| < 2$	Original inequality
$-2 < x - 5 < 2$	Equivalent inequalities
$-2 + 5 < x - 5 + 5 < 2 + 5$	Add 5 to all three parts.
$3 < x < 7$	Simplify.

The solution set is all real numbers that are greater than 3 and less than 7, which is denoted by $(3, 7)$. The graph is shown in Figure P.9.



$|x - 5| < 2$

FIGURE P.9

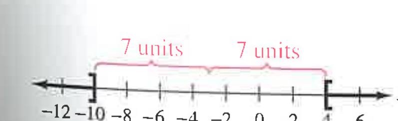
EXAMPLE 6 *Solving an Absolute Value Inequality*

Solve $|x + 3| \ge 7$.

Solution

$|x + 3| \ge 7$			Original inequality
$x + 3 \le -7$	or	$x + 3 \ge 7$	Equivalent inequalities
$x + 3 - 3 \le -7 - 3$		$x + 3 - 3 \ge 7 - 3$	Subtract 3 from both sides.
$x \le -10$		$x \ge 4$	Simplify.

The solution set is all real numbers that are less than or equal to -10 *or* greater than or equal to 4, which is denoted by $(-\infty, -10] \cup [4, \infty)$. The graph is shown in Figure P.10.



$|x + 3| \ge 7$

FIGURE P.10

Other Types of Inequalities

To solve a polynomial inequality, you can use the fact that a polynomial can change signs only at its zeros (the x-values that make the polynomial equal to zero). Between two consecutive zeros a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **critical numbers** of the inequality, and the resulting intervals are the **test intervals** for the inequality.

EXAMPLE 7 *Solving a Polynomial Inequality*

Solve $x^2 - x - 6 < 0$.

Solution

By factoring the quadratic as $x^2 - x - 6 = (x + 2)(x - 3)$, you can see that the critical numbers are $x = -2$ and $x = 3$. Thus, the polynomial's test intervals are

$(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. Test intervals

In each test interval, choose a representative x-value and evaluate the polynomial.

Interval	*x-Value*	*Polynomial Value*	*Conclusion*
$(-\infty, -2)$	$x = -3$	$(-3)^2 - (-3) - 6 = 6$	Positive
$(-2, 3)$	$x = 0$	$(0)^2 - (0) - 6 = -6$	Negative
$(3, \infty)$	$x = 4$	$(4)^2 - (4) - 6 = 6$	Positive

From this, you can conclude that the polynomial is positive for all x-values in $(-\infty, -2)$ and $(3, \infty)$, and is negative for all x-values in $(-2, 3)$. This implies that the solution of the inequality $x^2 - x - 6 < 0$ is the interval $(-2, 3)$, as shown in Figure P.11.

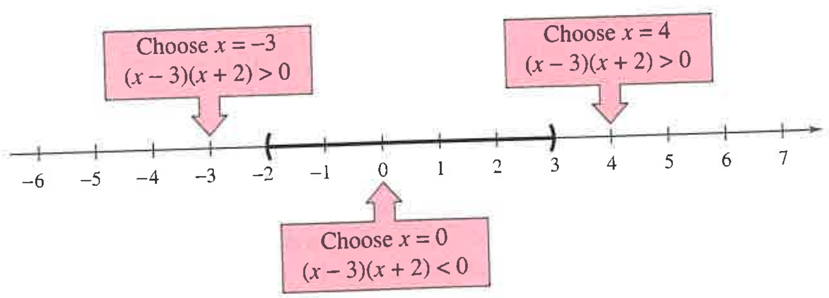


FIGURE P.11

Study Tip

As with linear inequalities, you can check the reasonableness of a solution by substituting x-values into the original inequality. For instance, to check the solution found in Example 7, try substituting several x-values from the interval $(-2, 3)$ into the inequality

$$x^2 - x - 6 < 0.$$

Regardless of which x-values you choose, the inequality should be satisfied.

The concepts of critical numbers and test intervals can be extended to rational inequalities. To do this, use the fact that the value of a rational expression can change sign only at its *zeros* (the x-values for which its numerator is zero) and its *undefined values* (the x-values for which its denominator is zero). These two types of numbers make up the **critical numbers** of a rational inequality.

EXAMPLE 8 *Solving a Rational Inequality*

Solve $\dfrac{2x - 7}{x - 5} \le 3$.

Solution

$$\frac{2x - 7}{x - 5} \le 3 \qquad \text{Original inequality}$$

$$\frac{2x - 7}{x - 5} - 3 \le 0 \qquad \text{Standard form}$$

$$\frac{2x - 7 - 3x + 15}{x - 5} \le 0 \qquad \text{Add fractions.}$$

$$\frac{-x + 8}{x - 5} \le 0 \qquad \text{Simplify.}$$

Critical Numbers: $x = 5, x = 8$

Test Intervals: $(-\infty, 5), (5, 8), (8, \infty)$

Test: Is $\dfrac{-x + 8}{x - 5} \le 0$?

After testing these intervals, as shown in Figure P.12, you can see that the rational expression $(-x + 8)/(x - 5)$ is negative in the open intervals $(-\infty, 5)$ and $(8, \infty)$. Moreover, because $(-x + 8)/(x - 5) = 0$ when $x = 8$, you can conclude that the solution set consists of all real numbers in the intervals

$(-\infty, 5) \cup [8, \infty)$. Solution set

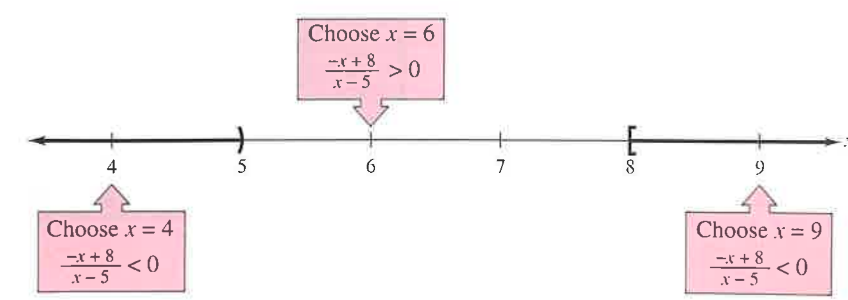


FIGURE P.12

EXAMPLE 9 *Finding the Domain of an Expression*

Find the domain of $\sqrt{64 - 4x^2}$.

Solution

Remember that the domain of an expression is the set of all x-values for which the expression is defined. Because $\sqrt{64 - 4x^2}$ is defined (has real values) only if $64 - 4x^2$ is nonnegative, the domain is given by $64 - 4x^2 \geq 0$.

$64 - 4x^2 \geq 0$	Standard form
$16 - x^2 \geq 0$	Divide both sides by 4.
$(4 - x)(4 + x) \geq 0$	Factored form

Thus, the inequality has two critical numbers: -4 and 4. You can use these two numbers to test the inequality as follows.

Critical Numbers: $x = -4, x = 4$

Test Intervals: $(-\infty, -4), (-4, 4), (4, \infty)$

Test: Is $(4 - x)(4 + x) \geq 0$?

A test shows that $64 - 4x^2$ is greater than or equal to 0 in the *closed interval* $[-4, 4]$. Thus, the domain of the expression $\sqrt{64 - 4x^2}$ is the interval $[-4, 4]$, as shown in Figure P.13.

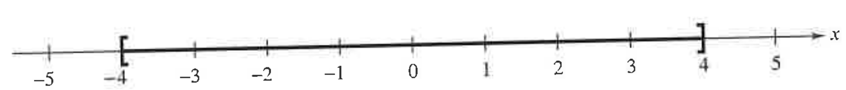


FIGURE P.13

GROUP ACTIVITY

COMMUNICATING MATHEMATICALLY

Four different properties of inequalities are listed on page 66. For each property, (a) translate the mathematical statement into a *verbal statement*, (b) compile a list of several numerical examples that demonstrate the property, and (c) construct a number line or series of number lines that graphically illustrates the property.

WARM UP

Determine which of the two numbers is larger.

1. $-\frac{1}{2}, -7$
2. $-\frac{1}{3}, -\frac{1}{6}$
3. $-\pi, -3$
4. $-6, \frac{13}{2}$

Use inequality notation to describe the statement.

5. x is nonnegative.
6. z is strictly between -3 and 10.
7. P is no more than 2.
8. W is at least 200.

Evaluate the expression for the given values of x.

9. $|x - 10|, \quad x = 12, x = 3$
10. $|2x - 3|, \quad x = \frac{3}{2}, x = 1$

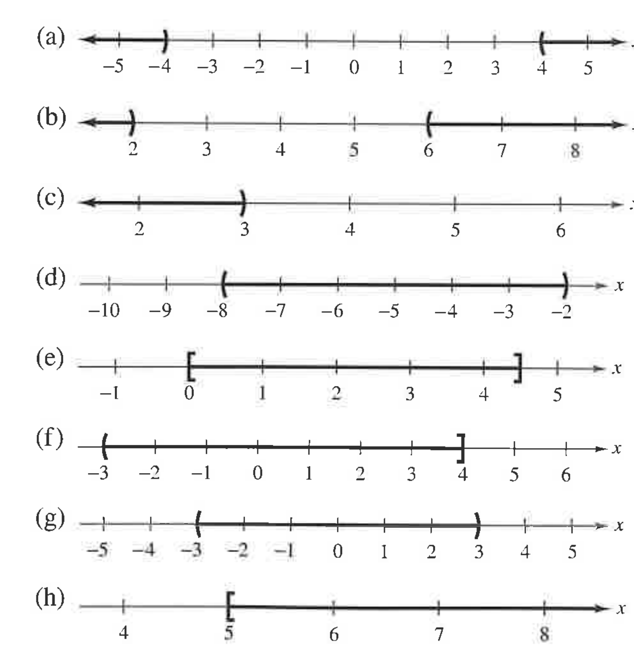
P.6 Exercises

In Exercises 1–4, write an inequality to represent the interval, and state whether the interval is bounded or unbounded.

1. $[-1, 3]$
2. $(4, 10]$
3. $(10, \infty)$
4. $[-6, \infty)$

In Exercises 5–12, match the inequality with its graph. Then write the inequality in interval form. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

5. $x < 3$
6. $x \geq 5$
7. $-3 < x \leq 4$
8. $0 \leq x \leq \frac{9}{2}$
9. $|x| < 3$
10. $|x| > 4$
11. $|x - 4| > 2$
12. $|x + 5| < 3$



In Exercises 13–20, determine whether the values of x are solutions of the inequality.

Inequality	*Values*	
13. $5x - 12 > 0$	(a) $x = 3$	(b) $x = -3$
	(c) $x = \frac{5}{2}$	(d) $x = \frac{3}{2}$
14. $x + 1 < \dfrac{2x}{3}$	(a) $x = 0$	(b) $x = 4$
	(c) $x = -4$	(d) $x = -3$
15. $0 < \dfrac{x-2}{4} < 2$	(a) $x = 4$	(b) $x = 10$
	(c) $x = 0$	(d) $x = \frac{7}{2}$
16. $\lvert 2x - 3\rvert < 15$	(a) $x = -6$	(b) $x = 0$
	(c) $x = 12$	(d) $x = 7$
17. $x^2 - 3 < 0$	(a) $x = 3$	(b) $x = 0$
	(c) $x = \frac{3}{2}$	(d) $x = -5$
18. $x^2 - x - 12 \geq 0$	(a) $x = 5$	(b) $x = 0$
	(c) $x = -4$	(d) $x = -3$
19. $\dfrac{x+2}{x-4} \geq 3$	(a) $x = 5$	(b) $x = 4$
	(c) $x = -\frac{9}{2}$	(d) $x = \frac{9}{2}$
20. $\dfrac{3x^2}{x^2+4} < 1$	(a) $x = -2$	(b) $x = -1$
	(c) $x = 0$	(d) $x = 3$

In Exercises 21–52, solve the inequality and sketch the solution on the real number line.

21. $4x < 12$

22. $2x > 3$

23. $-10x < 40$

24. $-6x > 15$

25. $x - 5 \geq 7$

26. $x + 7 \leq 12$

27. $4(x + 1) < 2x + 3$

28. $2x + 7 < 3$

29. $4 - 2x < 3$

30. $6x - 4 \leq 2$

31. $1 < 2x + 3 < 9$

32. $-8 \leq 1 - 3(x - 2) < 13$

33. $-4 < \dfrac{2x-3}{3} < 4$

34. $0 \leq \dfrac{x+3}{2} < 5$

35. $\dfrac{3}{4} > x + 1 > \dfrac{1}{4}$

36. $-1 < -\dfrac{x}{3} < 1$

37. $\lvert x\rvert < 5$

38. $\lvert 2x\rvert < 6$

39. $\left\lvert \dfrac{x}{2}\right\rvert > 3$

40. $\lvert 5x\rvert > 10$

41. $\lvert x - 20\rvert \leq 4$

42. $\lvert x - 7\rvert < 6$

43. $\lvert x - 20\rvert \geq 4$

44. $\lvert x + 14\rvert + 3 > 17$

45. $\left\lvert \dfrac{x-3}{2}\right\rvert \geq 5$

46. $\lvert 1 - 2x\rvert < 5$

47. $\lvert 9 - 2x\rvert - 2 < -1$

48. $\left\lvert 1 - \dfrac{2x}{3}\right\rvert < 1$

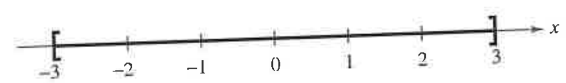
49. $2\lvert x + 10\rvert \geq 9$

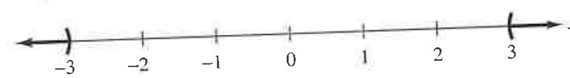
50. $3\lvert 4 - 5x\rvert \leq 9$

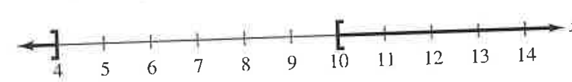
51. $\lvert x - 5\rvert < 0$

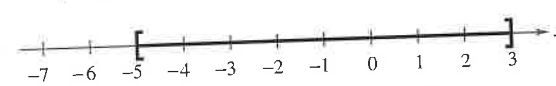
52. $\lvert x - 5\rvert \geq 0$

In Exercises 53–60, use absolute value notation to define each interval (or pair of intervals) on the real number line.

53. 

54. 

55. 

56. 

57. All real numbers within 10 units of 12

58. All real numbers at least five units from 8

59. All real numbers more than five units from -3

60. All real numbers no more than seven units from -6

61. ***Think About It*** The graph of $\lvert x - 5\rvert < 3$ can be described as *all real numbers within three units of 5*. Give a similar description of $\lvert x - 10\rvert < 8$.

62. ***Think About It*** The graph of $\lvert x - 2\rvert > 5$ can be described as *all real numbers more than five units from 2*. Give a similar description of $\lvert x - 8\rvert > 4$.

In Exercises 63–84, solve the inequality and graph the solution on the real number line.

63. $x^2 \leq 9$

64. $x^2 < 5$

65. $x^2 > 4$

66. $(x - 3)^2 \geq 1$

67. $(x + 2)^2 < 25$

68. $(x + 6)^2 \leq 8$

69. $x^2 + 4x + 4 \geq 9$

70. $x^2 - 6x + 9 < 16$

71. $3(x - 1)(x + 1) > 0$

72. $6(x + 2)(x - 1) < 0$

73. $x^2 + 2x - 3 < 0$

74. $x^2 - 4x - 1 > 0$

75. $4x^3 - 6x^2 < 0$

76. $4x^3 - 12x^2 > 0$

77. $(x - 1)^2(x + 2)^3 \geq 0$

78. $x^4(x - 3) \leq 0$

79. $\dfrac{1}{x} - x > 0$

80. $\dfrac{1}{x} - 4 < 0$

81. $\dfrac{x+6}{x+1} - 2 < 0$

82. $\dfrac{x+12}{x+2} - 3 \geq 0$

83. $\dfrac{4}{x+5} > \dfrac{1}{2x+3}$

84. $\dfrac{5}{x-6} > \dfrac{3}{x+2}$

In Exercises 85–90, find the interval(s) on the real number line for which the radicand is nonnegative (greater than or equal to zero).

85. $\sqrt{x - 5}$

86. $\sqrt[4]{6x + 15}$

87. $\sqrt[4]{4 - x^2}$

88. $\sqrt{x^2 - 4}$

89. $\sqrt{x^2 - 7x + 12}$

90. $\sqrt{144 - 9x^2}$

91. ***Simple Interest*** In order for an investment of \$1000 to grow to *more than* \$1250 in 2 years, what must the annual interest rate be? [$A = P(1 + rt)$]

92. ***Comparative Shopping*** You can rent a midsize car from Company A for \$250 per week with unlimited mileage. A similar car can be rented from Company B for \$150 per week, plus \$0.25 cents for each mile driven. How many miles must you drive in a week to make the rental fee of Company B *greater than* that of Company A?

93. ***Break-Even Analysis*** The revenue for selling x units of a product is

$$R = 115.95x.$$

The cost of producing x units is

$$C = 95x + 750.$$

To obtain a profit, the revenue must be *greater than* the cost. For what values of x will this product return a profit?

94. ***Annual Operating Cost*** A utility company has a fleet of vans. The annual operating cost per van is

$$C = 0.32m + 2300$$

where m is the number of miles traveled by a van in a year. What number of miles will yield an annual operating cost that is less than \$10,000?

95. ***Relative Humidity*** A certain electronic device is to be operated in an environment with relative humidity h in the interval defined by

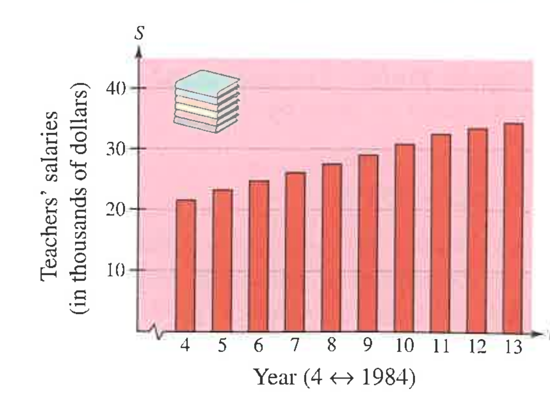
$$\lvert h - 50\rvert \leq 30.$$

What are the minimum and maximum relative humidities for the operation of this device?

96. ***Teachers' Salaries*** The average salary S (in thousands of dollars) for elementary and secondary teachers in the United States from 1984 to 1993 is approximated by the model

$$S = 15.812 + 1.472t$$

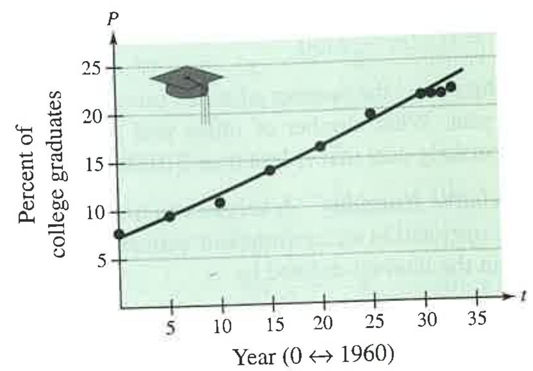
where $t = 4$ represents 1984 (see figure). According to this model, when will the average salary *exceed* \$40,000? (Source: National Education Association)



97. ***Percent of College Graduates*** The percent P of the American population that graduated from college from 1960 to 1993 is approximated by the model

$$P = 7.34 + 0.41t + 0.002t^2$$

where the time t represents the calendar year, with $t = 0$ corresponding to 1960 (see figure). According to this model, when will the percent of college graduates exceed 25% of the population? (Source: U.S. Bureau of the Census)



98. ***Accuracy of Measurement*** The side of a square is measured as 10.4 inches with a possible error of $\frac{1}{16}$ inch. Using these measurements, determine the interval containing the area of the square.

99. ***Exploration*** Find sets of values of a, b, and c such that $0 \le x \le 10$ is a solution of the inequality $|ax - b| \le c$.

100. ***Geometry*** A rectangular playing field with a perimeter of 100 meters is to have an area of at least 500 square meters. Within what bounds must the length of the rectangle lie?

101. ***Company Profits*** The revenue and cost equations for a product are given by

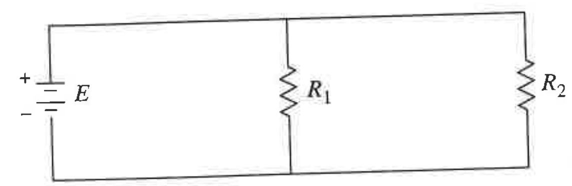
$$R = x(50 - 0.0002x)$$
$$C = 12x + 150{,}000$$

where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least $1,650,000?

102. ***Resistors*** When two resistors of resistances R_1 and R_2 are connected in parallel (see figure), the total resistance R satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Find R_1 for a parallel circuit in which $R_2 = 2$ ohms and R must be at least 1 ohm.



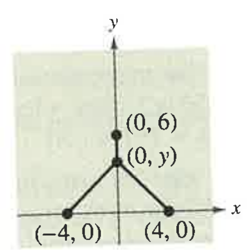
103. ***Power Supply*** Two factories are located at the coordinates $(-4, 0)$ and $(4, 0)$ and their power supply located at the point $(0, 6)$ (see figure).

(a) Write an equation, in terms of y, giving the amount L of power line required to supply both factories.

(b) Determine the interval of values for y in the context of the problem. Determine L for the two endpoints of the interval. Will L increase or decrease for values of y not at the endpoints of the interval?

(c) Use a graphing utility to graph the equation in part (a) and use the graph to verify your answers in part (b).

(d) Find the values of y such that $L < 13$.



P.7 Errors and the Algebra of Calculus

See the Group Activity on page 82 for an example of how algebra is used in calculus.

Algebraic Errors to Avoid □ *Some Algebra of Calculus*

Algebraic Errors to Avoid

This section contains five lists of common algebraic errors: errors involving parentheses, errors involving fractions, errors involving exponents, errors involving radicals, and errors involving cancellation. Many of these errors are made because they seem to be the *easiest* things to do.

ERRORS INVOLVING PARENTHESES

Potential Error	*Correct Form*	*Comment*
$a - (x - b) \ne a - x - b$	$a - (x - b) = a - x + b$	Change all signs when distributing minus sign.
$(a + b)^2 \ne a^2 + b^2$	$(a + b)^2 = a^2 + 2ab + b^2$	Remember the middle term when squaring binomials.
$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) \ne \frac{1}{2}(ab)$	$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{4}(ab) = \frac{ab}{4}$	$\frac{1}{2}$ occurs twice as a factor.
$(3x + 6)^2 \ne 3(x + 2)^2$	$(3x + 6)^2 = [3(x + 2)]^2 = 3^2(x + 2)^2$	When factoring, apply exponents to all factors.

ERRORS INVOLVING FRACTIONS

Potential Error	*Correct Form*	*Comment*
$\frac{a}{x + b} \ne \frac{a}{x} + \frac{a}{b}$	Leave as $\frac{a}{x + b}$.	Do not add denominators when adding fractions.
$\frac{\left(\frac{x}{a}\right)}{b} \ne \frac{bx}{a}$	$\frac{\left(\frac{x}{a}\right)}{b} = \left(\frac{x}{a}\right)\left(\frac{1}{b}\right) = \frac{x}{ab}$	Multiply by the reciprocal when dividing fractions.
$\frac{1}{a} + \frac{1}{b} \ne \frac{1}{a + b}$	$\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$	Use the property for adding fractions.
$\frac{1}{3x} \ne \frac{1}{3}x$	$\frac{1}{3x} = \frac{1}{3} \cdot \frac{1}{x}$	Use the property for multiplying fractions.
$(1/3)x \ne \frac{1}{3x}$	$(1/3)x = \frac{1}{3} \cdot x = \frac{x}{3}$	Be careful when using a slash to denote division.
$(1/x) + 2 \ne \frac{1}{x + 2}$	$(1/x) + 2 = \frac{1}{x} + 2 = \frac{1 + 2x}{x}$	Be careful when using a slash to denote division.

Errors Involving Exponents

Potential Error	Correct Form	Comment
$(x^2)^3 \neq x^5$	$(x^2)^3 = x^{2 \cdot 3} = x^6$	Multiply exponents when raising a power to a power.
$x^2 \cdot x^3 \neq x^6$	$x^2 \cdot x^3 = x^{2+3} = x^5$	Add exponents when multiplying powers with like bases.
$2x^3 \neq (2x)^3$	$2x^3 = 2(x^3)$	Exponents have priority over coefficients.
$\frac{1}{x^2 - x^3} \neq x^{-2} - x^{-3}$	Leave as $\frac{1}{x^2 - x^3}$.	Do not move term-by-term from denominator to numerator.

Errors Involving Radicals

Potential Error	Correct Form	Comment
$\sqrt{5x} \neq 5\sqrt{x}$	$\sqrt{5x} = \sqrt{5}\sqrt{x}$	Radicals apply to every factor inside the radical.
$\sqrt{x^2 + a^2} \neq x + a$	Leave as $\sqrt{x^2 + a^2}$.	Do not apply radicals term-by-term.
$\sqrt{-x + a} \neq -\sqrt{x - a}$	Leave as $\sqrt{-x + a}$.	Do not factor minus signs out of square roots.

Errors Involving Cancellation

Potential Error	Correct Form	Comment
$\frac{a + bx}{a} \neq 1 + bx$	$\frac{a + bx}{a} = \frac{a}{a} + \frac{bx}{a} = 1 + \frac{b}{a}x$	Cancel common factors, not common terms.
$\frac{a + ax}{a} \neq a + x$	$\frac{a + ax}{a} = \frac{a(1 + x)}{a} = 1 + x$	Factor before canceling.
$1 + \frac{x}{2x} \neq 1 + \frac{1}{x}$	$1 + \frac{x}{2x} = 1 + \frac{1}{2} = \frac{3}{2}$	Cancel common factors.

For many people, a good way to avoid errors is to *work slowly, write neatly,* and *talk to yourself.* Each time you write a step, ask yourself why the step is algebraically legitimate. For instance, when you write

$$\frac{2x}{6} = \frac{\cancel{2} \cdot x}{\cancel{2} \cdot 3} = \frac{x}{3}$$

you can justify your work because *dividing the numerator and denominator by the same nonzero number produces an equivalent fraction.*

Some Algebra of Calculus

In calculus it is often necessary to take a simplified algebraic expression and "unsimplify" it. See the following list, taken from a standard calculus text.

Unusual Factoring

Expression	Useful Calculus Form	Comment
$\frac{5x^4}{8}$	$\frac{5}{8}x^4$	Write with fractional coefficient.
$\frac{x^2 + 3x}{-6}$	$-\frac{1}{6}(x^2 + 3x)$	Write with fractional coefficient.
$2x^2 - x - 3$	$2\left(x^2 - \frac{x}{2} - \frac{3}{2}\right)$	Factor out the leading coefficient.
$\frac{x}{2}(x + 1)^{-1/2} + (x + 1)^{1/2}$	$\frac{(x + 1)^{-1/2}}{2}[x + 2(x + 1)]$	Factor out factor with least power.

Inserting Factors and Terms

Expression	Useful Calculus Form	Comment
$(2x - 1)^3$	$\frac{1}{2}(2x - 1)^3(2)$	Multiply and divide by 2.
$7x^2(4x^3 - 5)^{1/2}$	$\frac{7}{12}(4x^3 - 5)^{1/2}(12x^2)$	Multiply and divide by 12.
$\frac{4x^2}{9} - 4y^2 = 1$	$\frac{x^2}{9/4} - \frac{y^2}{1/4} = 1$	Write with fractional denominators.
$\frac{x}{x + 1}$	$\frac{x + 1 - 1}{x + 1} = 1 - \frac{1}{x + 1}$	Add and subtract the same term.

Writing with Negative Exponents

Expression	Useful Calculus Form	Comment
$\frac{9}{5x^3}$	$\frac{9}{5}x^{-3}$	Move the factor to the numerator and change the sign of the exponent.
$\frac{7}{\sqrt{2x - 3}}$	$7(2x - 3)^{-1/2}$	Move the factor to the numerator and change the sign of the exponent.

WRITING A FRACTION AS A SUM

Expression	*Useful Calculus Form*	*Comment*
$\dfrac{x+2x^2+1}{\sqrt{x}}$	$x^{1/2}+2x^{3/2}+x^{-1/2}$	Divide each term by $x^{1/2}$.
$\dfrac{1+x}{x^2+1}$	$\dfrac{1}{x^2+1}+\dfrac{x}{x^2+1}$	Rewrite the fraction as the sum of fractions.
$\dfrac{2x}{x^2+2x+1}$	$\dfrac{2x+2-2}{x^2+2x+1}$	Add and subtract the same term.
	$=\dfrac{2x+2}{x^2+2x+1}-\dfrac{2}{(x+1)^2}$	Rewrite the fraction as the difference of fractions.
$\dfrac{x^2-2}{x+1}$	$x-1-\dfrac{1}{x+1}$	Use long division. (See Section 2.3.)
$\dfrac{x+7}{x^2-x-6}$	$\dfrac{2}{x-3}-\dfrac{1}{x+2}$	Use the method of partial fractions. (See Section 2.8.)

The next four examples demonstrate many of the steps in the preceding lists.

EXAMPLE 1 *Factors Involving Negative Exponents*

Factor $x(x+1)^{-1/2}+(x+1)^{1/2}$.

Solution

When multiplying factors with like bases, you add exponents. When factoring, you are undoing multiplication, and so you *subtract* exponents.

$$\begin{aligned}x(x+1)^{-1/2}+(x+1)^{1/2} &= (x+1)^{-1/2}[x(x+1)^0+(x+1)^1]\\ &= (x+1)^{-1/2}[x+(x+1)]\\ &= (x+1)^{-1/2}(2x+1)\end{aligned}$$

Here is another way to factor the expression in Example 1.

$$\begin{aligned}x(x+1)^{-1/2}+(x+1)^{1/2} &= x(x+1)^{-1/2}+(x+1)^{1/2}\cdot\frac{(x+1)^{1/2}}{(x+1)^{1/2}}\\ &= \frac{x(x+1)^0+(x+1)^1}{(x+1)^{1/2}}\\ &= \frac{2x+1}{\sqrt{x+1}}\end{aligned}$$

EXAMPLE 2 *Rewriting Fractions*

Explain the following.

$$\frac{4x^2}{9}-4y^2=\frac{x^2}{9/4}-\frac{y^2}{1/4}$$

Solution

To write the expression on the left side of the equation in the form given on the right, multiply the numerators and denominators of both terms by $\frac{1}{4}$.

$$\begin{aligned}\frac{4x^2}{9}-4y^2 &= \frac{4x^2}{9}\left(\frac{1/4}{1/4}\right)-4y^2\left(\frac{1/4}{1/4}\right)\\ &= \frac{x^2}{9/4}-\frac{y^2}{1/4}\end{aligned}$$

EXAMPLE 3 *Rewriting with Negative Exponents*

Rewrite the expression using negative exponents.

$$\frac{2}{5x^3}-\frac{1}{\sqrt{x}}+\frac{3}{5(4x)^2}$$

Solution

Begin by writing the second term in exponential form.

$$\begin{aligned}\frac{2}{5x^3}-\frac{1}{\sqrt{x}}+\frac{3}{5(4x)^2} &= \frac{2}{5x^3}-\frac{1}{x^{1/2}}+\frac{3}{5(4x)^2}\\ &= \frac{2}{5}x^{-3}-x^{-1/2}+\frac{3}{5}(4x)^{-2}\end{aligned}$$

EXAMPLE 4 *Writing a Fraction as a Sum of Terms*

Rewrite the fraction as the sum of three terms.

$$\frac{x+2x^2+1}{\sqrt{x}}$$

Solution

$$\begin{aligned}\frac{x+2x^2+1}{\sqrt{x}} &= \frac{x}{x^{1/2}}+\frac{2x^2}{x^{1/2}}+\frac{1}{x^{1/2}}\\ &= x^{1/2}+2x^{3/2}+x^{-1/2}\end{aligned}$$

GROUP ACTIVITY

ALGEBRA AND CALCULUS

Suppose you are taking a course in calculus, and for one of the homework problems you obtain the following answer.

$$\frac{1}{10}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}$$

The answer in the back of the book is given as follows.

$$\frac{1}{15}(2x-1)^{3/2}(3x+1)$$

Are these two answers equivalent? If so, show how the second answer can be obtained from the first. Then, use the same technique to simplify the following expressions.

a. $\frac{2}{3}x(2x-3)^{3/2}-\frac{2}{15}(2x-3)^{5/2}$

b. $\frac{2}{3}x(4+x)^{3/2}-\frac{2}{15}(4+x)^{5/2}$

WARM UP

Factor the expression.

1. a^3-16a
2. u^3+125v^3
3. $2+5x-12x^2$
4. $z^3+3z^2-4z-12$

Perform the operations and simplify.

5. $\frac{8-z}{4z^3}\cdot\frac{8z}{z-8}$
6. $\frac{x^2-y^2}{2x^2-8x}\div\frac{(x-y)^2}{2xy}$
7. $\frac{1}{x}-\frac{3}{y}+\frac{3x-y}{xy}$
8. $\frac{5}{x-2}-\frac{4}{2-x}$
9. $\dfrac{\left(16-\frac{1}{x^2}\right)}{\left(\frac{1}{4x^2}-4\right)}$
10. $\dfrac{\left(\frac{1}{2+h}-\frac{1}{2}\right)}{h}$

P.7 Exercises

In Exercises 1–24, find and correct any errors.

1. $2x-(3y+4)=2x-3y+4$
2. $\frac{4}{16x-(2x+1)}=\frac{4}{14x+1}$
3. $5z+3(x-2)=5z+3x-2$
4. $\frac{x-1}{(5-x)(-x)}=\frac{1-x}{x(5-x)}$
5. $-\frac{x-3}{x-1}=\frac{3-x}{1-x}$
6. $x(yz)=(xy)(xz)$
7. $a\left(\frac{x}{y}\right)=\frac{ax}{ay}$
8. $(5z)(6z)=30z$
9. $(4x)^2=4x^2$
10. $\left(\frac{x}{y}\right)^3=\frac{x^3}{y}$
11. $\sqrt{x+9}=\sqrt{x}+3$
12. $\sqrt{25-x^2}=5-x$
13. $\frac{6x+y}{6x-y}=\frac{x+y}{x-y}$
14. $\frac{2x^2+1}{5x}=\frac{2x+1}{5}$
15. $\frac{1}{x+y^{-1}}=\frac{y}{x+1}$
16. $\frac{1}{a^{-1}+b^{-1}}=\left(\frac{1}{a+b}\right)^{-1}$
17. $x(2x-1)^2=(2x^2-x)^2$
18. $x(x+5)^{1/2}=(x^2+5x)^{1/2}$
19. $\sqrt[3]{x^3+7x^2}=x^2\sqrt[3]{x+7}$
20. $(3x^2-6x)^3=3x(x-2)^3$
21. $\frac{3}{x}+\frac{4}{y}=\frac{7}{x+y}$
22. $\frac{7+5(x+3)}{x+3}=12$
23. $\frac{1}{2y}=(1/2)y$
24. $\frac{2x+3x^2}{4x}=\frac{2+3x^2}{4}$

In Exercises 25–52, insert the required factor in the parentheses.

25. $\frac{3x+2}{5}=\frac{1}{5}(\quad)$
26. $\frac{7x^2}{10}=\frac{7}{10}(\quad)$
27. $\frac{2}{3}x^2+\frac{1}{3}x+5=\frac{1}{3}(\quad)$
28. $\frac{3}{4}x+\frac{1}{2}=\frac{1}{4}(\quad)$
29. $\frac{1}{3}x^3+5=(\quad)(x^3+15)$
30. $\frac{5}{2}z^2-\frac{1}{4}z+2=(\quad)(10z^2-z+8)$
31. $x(2x^2+15)=(\quad)(2x^2+15)(2x)$
32. $x^2(x^3-1)^4=(\quad)(x^3-1)^4(3x^2)$
33. $x(1-2x^2)^3=(\quad)(1-2x^2)^3(-4x)$
34. $5x\sqrt[3]{1+x^2}=(\quad)\sqrt[3]{1+x^2}(2x)$
35. $\frac{1}{\sqrt{x}(1+\sqrt{x})^2}=(\quad)\frac{1}{(1+\sqrt{x})^2}\left(\frac{1}{2\sqrt{x}}\right)$
36. $\frac{4x+6}{(x^2+3x+7)^3}=(\quad)\frac{1}{(x^2+3x+7)^3}(2x+3)$
37. $\frac{x+1}{(x^2+2x-3)^2}=(\quad)\frac{1}{(x^2+2x-3)^2}(2x+2)$
38. $\frac{1}{(x-1)\sqrt{(x-1)^4-4}}=\frac{(\quad)}{(x-1)^2\sqrt{(x-1)^4-4}}$
39. $\frac{3}{x}+\frac{5}{2x^2}-\frac{3}{2}x=(\quad)(6x+5-3x^3)$
40. $\frac{(x-1)^2}{169}+(y+5)^2=\frac{(x-1)^3}{169(\quad)}+(y+5)^2$
41. $\frac{9x^2}{25}+\frac{16y^2}{49}=\frac{x^2}{(\quad)}+\frac{y^2}{(\quad)}$
42. $\frac{3x^2}{4}-\frac{9y^2}{16}=\frac{x^2}{(\quad)}-\frac{y^2}{(\quad)}$
43. $\frac{x^2}{1/12}-\frac{y^2}{2/3}=\frac{12x^2}{(\quad)}-\frac{3y^2}{(\quad)}$
44. $\frac{x^2}{4/9}+\frac{y^2}{7/8}=\frac{9x^2}{(\quad)}+\frac{8y^2}{(\quad)}$

45. $\sqrt{x} + (\sqrt{x})^3 = \sqrt{x}(\quad)$

46. $x^{1/3} - 5x^{4/3} = x^{1/3}(\quad)$

47. $3(2x + 1)x^{1/2} + 4x^{3/2} = x^{1/2}(\quad)$

48. $(1 - 3x)^{4/3} - 4x(1 - 3x)^{1/3} = (1 - 3x)^{1/3}(\quad)$

49. $\dfrac{x^2}{\sqrt{x^2+1}} - \sqrt{x^2+1} = \dfrac{1}{\sqrt{x^2+1}}(\quad)$

50. $\dfrac{1}{2\sqrt{x}} + 5x^{3/2} - 10x^{5/2} = \dfrac{1}{2\sqrt{x}}(\quad)$

51. $\dfrac{1}{10}(2x+1)^{5/2} - \dfrac{1}{6}(2x+1)^{3/2} = \dfrac{(2x+1)^{3/2}}{15}(\quad)$

52. $\dfrac{3}{7}(t+1)^{7/3} - \dfrac{3}{4}(t+1)^{4/3} = \dfrac{3(t+1)^{4/3}}{28}(\quad)$

In Exercises 53–58, write the fraction as a sum of two or more terms.

53. $\dfrac{16 - 5x - x^2}{x}$

54. $\dfrac{x^3 - 5x^2 + 4}{x^2}$

55. $\dfrac{4x^3 - 7x^2 + 1}{x^{1/3}}$

56. $\dfrac{2x^5 - 3x^3 + 5x - 1}{x^{3/2}}$

57. $\dfrac{3 - 5x^2 - x^4}{\sqrt{x}}$

58. $\dfrac{x^3 - 5x^4}{3x^2}$

In Exercises 59–66, simplify the expression.

59. $\dfrac{-2(x^2-3)^{-3}(2x)(x+1)^3 - 3(x+1)^2(x^2-3)^{-2}}{[(x+1)^3]^2}$

60. $\dfrac{x^5(-3)(x^2+1)^{-4}(2x) - (x^2+1)^{-3}(5)x^4}{(x^5)^2}$

61. $\dfrac{(6x+1)^3(27x^2+2) - (9x^3+2x)(3)(6x+1)^2(6)}{[(6x+1)^3]^2}$

62. $\dfrac{(4x^2+9)^{1/2}(2) - (2x+3)\left(\frac{1}{2}\right)(4x^2+9)^{-1/2}(8x)}{[(4x^2+9)^{1/2}]^2}$

63. $\dfrac{(x+2)^{3/4}(x+3)^{-2/3} - (x+3)^{1/3}(x+2)^{-1/4}}{[(x+2)^{3/4}]^2}$

64. $\dfrac{\sqrt{2x-1} - \dfrac{x+2}{\sqrt{2x-1}}}{2x-1}$

65. $\dfrac{2(3x-1)^{1/3} - (2x+1)\left(\frac{1}{3}\right)(3x-1)^{-2/3}(3)}{(3x-1)^{2/3}}$

66. $\dfrac{(x+1)\left(\frac{1}{2}\right)(2x-3x^2)^{-1/2}(2-6x) - (2x-3x^2)^{1/2}}{(x+1)^2}$

67. (a) Verify that $y_1 = y_2$ analytically.

$$y_1 = x^2\left(\tfrac{1}{3}\right)(x^2+1)^{-2/3}(2x) + (x^2+1)^{1/3}(2x)$$

$$y_2 = \frac{2x(4x^2+3)}{3(x^2+1)^{2/3}}$$

(b) Complete the table and demonstrate the equality of part (a) numerically.

x	-2	-1	$-\frac{1}{2}$	0	1	2	$\frac{5}{2}$
y_1							
y_2							

68. (a) Verify that $y_1 = y_2$ analytically.

$$y_1 = -\frac{\sqrt{9-x^2}}{x^2} - \frac{1}{\sqrt{9-x^2}}$$

$$y_2 = \frac{-9}{x^2\sqrt{9-x^2}}$$

(b) Complete the table and demonstrate the equality of part (a) numerically.

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{4}$	1	2	$\frac{5}{2}$
y_1							
y_2							

69. ***Logical Reasoning*** Verify that $y_1 \neq y_2$ by letting $x = 0$ and evaluating y_1 and y_2 where

$$y_1 = 2x\sqrt{1-x^2} - \frac{x^3}{\sqrt{1-x^2}}$$

and

$$y_2 = \frac{2-3x^2}{\sqrt{1-x^2}}.$$

Change y_2 so that $y_1 = y_2$.

P.8 Graphical Representation of Data

The Cartesian Plane ▫ *The Distance Formula* ▫ *The Midpoint Formula* ▫ *Application*

See Example 2 on page 86 for an example of how to represent real-life data graphically.

The Cartesian Plane

Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane called the **rectangular coordinate system,** or the **Cartesian plane,** after the French mathematician René Descartes (1596–1650).

The Cartesian plane is formed by using two real lines intersecting at right angles, as shown in Figure P.14. The horizontal real line is usually called the ***x*-axis,** and the vertical real line is usually called the ***y*-axis.** The point of intersection of these two axes is the **origin,** and the two axes divide the plane into four parts called **quadrants.**

Each point in the plane corresponds to an **ordered pair** (x, y) of real numbers x and y, called **coordinates** of the point. The ***x*-coordinate** represents the directed distance from the y-axis to the point, and the ***y*-coordinate** represents the directed distance from the x-axis to the point, as shown in Figure P.15.

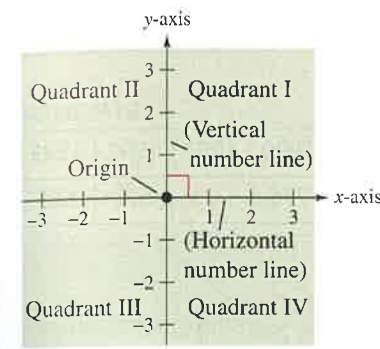


FIGURE P.14

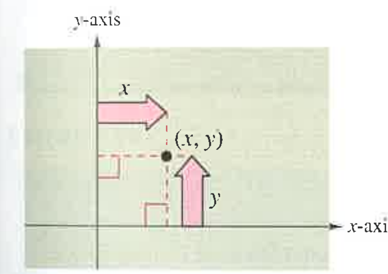


FIGURE P.15

(x, y)

Directed distance from y-axis | Directed distance from x-axis

NOTE The notation (x, y) denotes both a point in the plane and an open interval on the real line. The context will tell you which meaning is intended.

EXAMPLE 1 Plotting Points in the Cartesian Plane

Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$.

Solution

To plot the point

$$(-1, 2),$$

x-coordinate ⌟ ⌞ y-coordinate

imagine a vertical line through -1 on the x-axis and a horizontal line through 2 on the y-axis. The intersection of these two lines is the point $(-1, 2)$. The other four points can be plotted in a similar way, and are shown in Figure P.16.

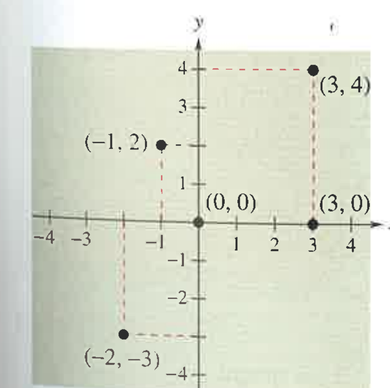


FIGURE P.16

NOTE In Example 2, you could have let $t = 1$ represent the year 1984. In that case, the horizontal axis would not have been broken, and the tick marks would have been labeled 1 through 10 (instead of 1984 through 1993).

The beauty of a rectangular coordinate system is that it allows you to see relationships between two variables. It would be difficult to overestimate the importance of Descartes's introduction of coordinates to the plane. Today, his ideas are in common use in virtually every scientific and business-related field.

EXAMPLE 2 Sketching a Scatter Plot

Real Life

From 1984 through 1993, the amount A (in millions of dollars) spent on fishing tackle in the United States is given in the table below, where t represents the year. Sketch a scatter plot of the data. (Source: National Sporting Goods Association)

t	1984	1985	1986	1987	1988	1989	1990	1991	1992	1993
A	616	681	773	830	766	769	776	711	678	685

Solution

To sketch a *scatter plot* of the data given in the table, you simply represent each pair of values by an ordered pair (t, A) and plot the resulting points, as shown in Figure P.17. For instance, the first pair of values is represented by the ordered pair (1984, 616). Note that the break in the t-axis indicates that the numbers between 0 and 1984 have been omitted.

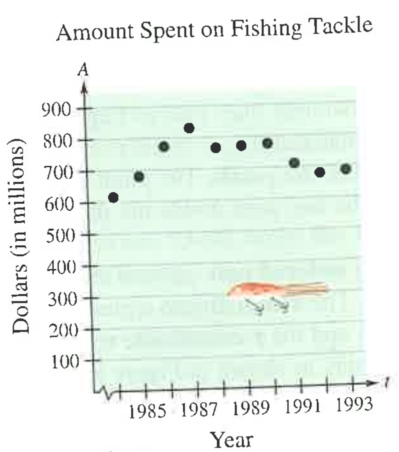
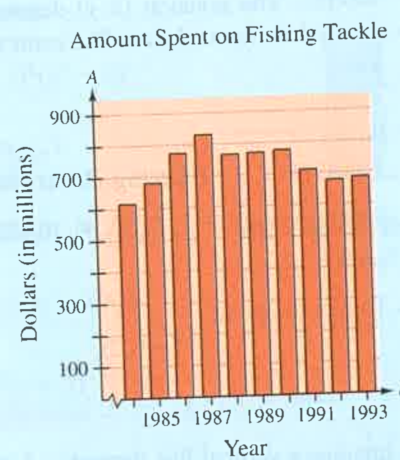


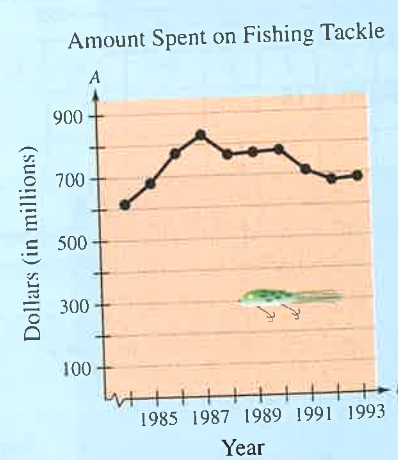
FIGURE P.17

TECHNOLOGY

The scatter plot in Example 2 is only one way to represent the data graphically. Two other techniques are shown at the right. The first is a *bar graph* and the second is a *line graph*. All three graphical representations were created with a computer. If you have access to a graphing utility, try using it to represent graphically the data given in Example 2.



Bar graph



Line graph

The Distance Formula

Recall from the Pythagorean Theorem that, for a right triangle with hypotenuse of length c and sides of lengths a and b, you have

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean Theorem}$$

as shown in Figure P.18. (The converse is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.)

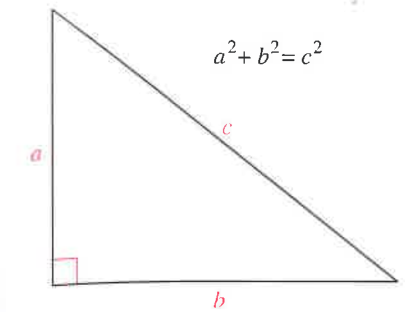


FIGURE P.18

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) in the plane. With these two points, a right triangle can be formed, as shown in Figure P.19. The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem, you can write

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This result is the **Distance Formula.**

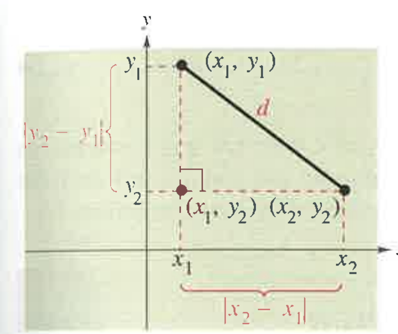


FIGURE P.19

> ***THE DISTANCE FORMULA***
>
> The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is
>
> $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 3 Finding a Distance

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

Solution

Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$. Then apply the Distance Formula as follows.

$$\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\
&= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} && \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\
&= \sqrt{(5)^2 + (3)^2} && \text{Simplify.} \\
&= \sqrt{34} \\
&\approx 5.83 && \text{Use a calculator.}
\end{aligned}$$

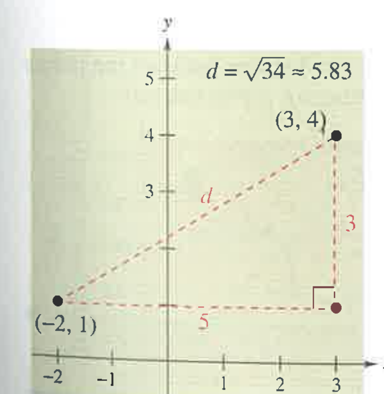


FIGURE P.20

Note in Figure P.20 that a distance of 5.83 looks about right.

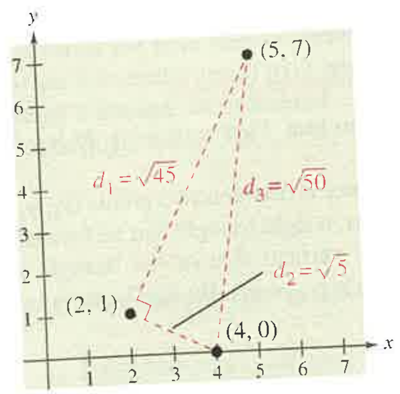


FIGURE P.21

EXAMPLE 4 Verifying a Right Triangle

Show that the points (2, 1), (4, 0), and (5, 7) are vertices of a right triangle.

Solution

The three points are plotted in Figure P.21. Using the Distance Formula, you can find the lengths of the three sides as follows.

$$d_1 = \sqrt{(5-2)^2 + (7-1)^2} = \sqrt{9+36} = \sqrt{45}$$
$$d_2 = \sqrt{(4-2)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5}$$
$$d_3 = \sqrt{(5-4)^2 + (7-0)^2} = \sqrt{1+49} = \sqrt{50}$$

Because

$$d_1^2 + d_2^2 = 45 + 5 = 50 = d_3^2,$$

you can conclude that the triangle must be a right triangle.

The figures provided with Examples 3 and 4 were not really essential to the solution. *Nevertheless,* we strongly recommend that you develop the habit of including sketches with your solutions—even if they are not required.

EXAMPLE 5 Finding the Length of a Pass

Real Life

A football quarterback throws a pass from the 5-yard line, 20 yards from the sideline. The pass is caught by a wide receiver on the 45-yard line, 50 yards from the same sideline, as shown in Figure P.22. How long is the pass?

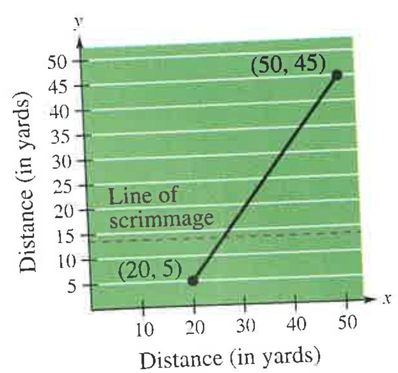


FIGURE P.22

Solution

You can find the length of the pass by finding the distance between the points (20, 5) and (50, 45).

$$\begin{aligned} d &= \sqrt{(50-20)^2 + (45-5)^2} && \text{Distance Formula} \\ &= \sqrt{900 + 1600} \\ &= 50 && \text{Simplify.} \end{aligned}$$

Thus, the pass is 50 yards long.

NOTE In Example 5, the scale along the goal line does not normally appear on a football field. However, when you use coordinate geometry to solve real-life problems, you are free to place the coordinate system in any way that is convenient to the solution of the problem.

THINK ABOUT THE PROOF

The Distance Formula can be used to prove the Midpoint Formula. Can you see how to do it? The details of the proof are listed in the appendix.

The Midpoint Formula

To find the **midpoint** of the line segment that joins two points in a coordinate plane, you can simply find the average values of the respective coordinates of the two endpoints.

THE MIDPOINT FORMULA

The midpoint of the segment joining the points (x_1, y_1) and (x_2, y_2) is

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

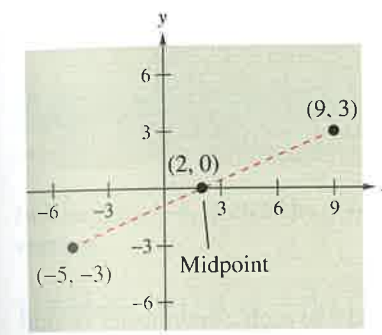


FIGURE P.23

EXAMPLE 6 Finding a Segment's Midpoint

Find the midpoint of the line segment joining the points (−5, −3) and (9, 3), as shown in Figure P.23.

Solution

Let $(x_1, y_1) = (-5, -3)$ and $(x_2, y_2) = (9, 3)$.

$$\begin{aligned} \text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) && \text{Midpoint Formula} \\ &= \left(\frac{-5+9}{2}, \frac{-3+3}{2}\right) && \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\ &= (2, 0) && \text{Simplify.} \end{aligned}$$

EXAMPLE 7 Estimating Annual Sales

Real Life

Ben and Jerry's had annual sales of \$132.0 million in 1992 and \$148.8 million in 1994. Without knowing any additional information, what would you estimate the 1993 sales to have been? (Source: Ben and Jerry's, Inc.)

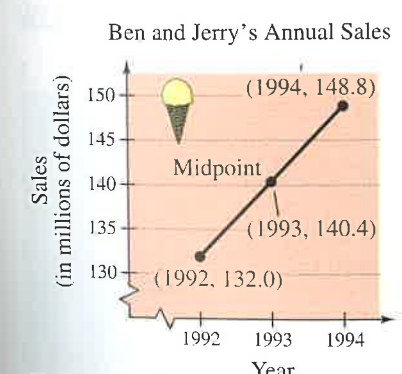


FIGURE P.24

Solution

One solution to the problem is to assume that sales followed a linear pattern. With this assumption, you can estimate the 1993 sales by finding the midpoint of the segment connecting the points (1992, 132.0) and (1994, 148.8).

$$\text{Midpoint} = \left(\frac{1992 + 1994}{2}, \frac{132.0 + 148.8}{2}\right) = (1993, 140.4)$$

Hence, you would estimate the 1993 sales to have been about \$140.4 million, as shown in Figure P.24. (The actual 1993 sales were \$140.3 million.)

Application

Much of computer graphics, including this computer-generated goldfish tesselation, consists of transformations of points in a coordinate plane. One type of transformation, a translation, is illustrated in Example 8. Other types include reflections, rotations, and stretches. *(Photo: Paul Morrell)*

EXAMPLE 8 *Translating Points in the Plane*

The triangle in Figure P.25(a) has vertices at the points $(-1, 2)$, $(1, -4)$, and $(2, 3)$. Shift the triangle three units to the right and two units up and find the vertices of the shifted triangle, as shown in Figure P.25(b).

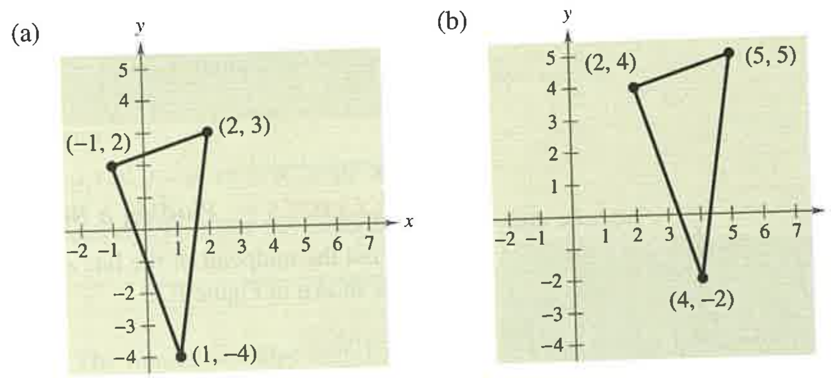


FIGURE P.25

Solution

To shift the vertices three units to the right, add 3 to each x-coordinate. To shift the vertices two units up, add 2 to each of the y-coordinates.

Original Point	*Translated Point*
$(-1, 2)$	$(-1 + 3, 2 + 2) = (2, 4)$
$(1, -4)$	$(1 + 3, -4 + 2) = (4, -2)$
$(2, 3)$	$(2 + 3, 3 + 2) = (5, 5)$

GROUP ACTIVITY

EXTENDING THE EXAMPLE

Example 8 shows how to translate points in a coordinate plane. How are the following transformed points related to the original points?

Original Point	*Transformed Point*
(x, y)	$(-x, y)$
(x, y)	$(x, -y)$
(x, y)	$(-x, -y)$

WARM UP

1. Find the distance between the real numbers -3.5 and 8.
2. Find the distance between the real numbers -20 and -7.

Simplify the expression.

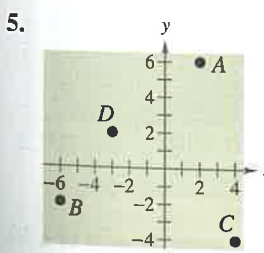
3. $\dfrac{4 + (-2)}{2}$
4. $\dfrac{-1 + (-3)}{2}$
5. $\dfrac{4.2 + 10.5}{2}$
6. $\dfrac{-5.4 - 3.2}{2}$
7. $\sqrt{(2 - 6)^2 + [1 - (-2)]^2}$
8. $\sqrt{(1 - 4)^2 + (-2 - 1)^2}$
9. $\sqrt{18} + \sqrt{45}$
10. $\sqrt{12} + \sqrt{44}$

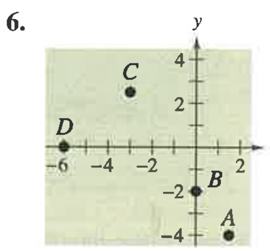
P.8 Exercises

In Exercises 1–4, sketch the polygon with the indicated vertices.

1. Triangle: $(-1, 1)$, $(2, -1)$, $(3, 4)$
2. Triangle: $(0, 3)$, $(-1, -2)$, $(4, 8)$
3. Square: $(2, 4)$, $(5, 1)$, $(2, -2)$, $(-1, 1)$
4. Parallelogram: $(5, 2)$, $(7, 0)$, $(1, -2)$, $(-1, 0)$

In Exercises 5 and 6, approximate the coordinates of the points.

5. 

6. 

In Exercises 7–10, find the coordinates of the point.

7. The point is located 3 units to the left of the y-axis and 4 units above the x-axis.
8. The point is located 8 units below the x-axis and 4 units to the right of the y-axis.
9. The point is located 5 units below the x-axis and the coordinates of the point are equal.
10. The point is on the x-axis and 12 units to the left of the y-axis.

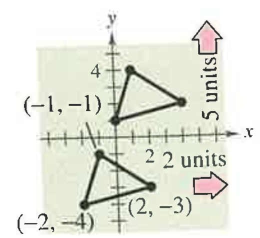
11. ***Think About It*** What is the y-coordinate of any point on the x-axis? What is the x-coordinate of any point on the y-axis?
12. ***Think About It*** When plotting points on the rectangular coordinate system, is it true that the scales on the x and y axes must be the same? Explain.

In Exercises 13–22, determine the quadrant(s) in which (x, y) is located so that the condition(s) is (are) satisfied.

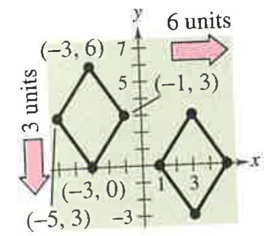
13. $x > 0$ and $y < 0$
14. $x < 0$ and $y < 0$
15. $x = -4$ and $y > 0$
16. $x > 2$ and $y = 3$
17. $y < -5$
18. $x > 4$
19. $(x, -y)$ is in the second quadrant.
20. $(-x, y)$ is in the fourth quadrant.
21. $xy > 0$
22. $xy < 0$

In Exercises 23 and 24, the polygon is shifted to a new position in the plane. Find the coordinates of the vertices of the polygon in its *new* position.

23.



24.



In Exercises 25 and 26, sketch a scatter plot of the data given in the table.

25. ***Normal Temperatures*** The normal temperature y (in degrees Fahrenheit) in Duluth, Minnesota, for each month x, where $x = 1$ represents January, is given in the table. (Source: NOAA)

x	1	2	3	4	5	6
y	6	12	23	38	50	59

x	7	8	9	10	11	12
y	65	63	54	44	28	14

26. ***Wal-Mart*** The number y of Wal-Mart stores for each year x from 1985 through 1994 is given in the table. (Source: Wal-Mart Annual Report for 1994)

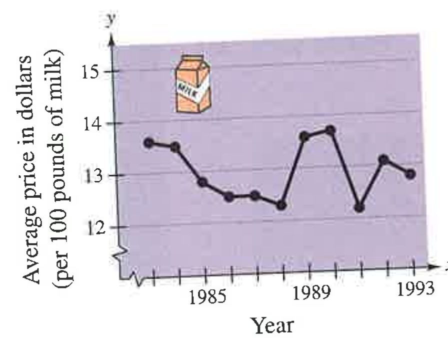
x	1985	1986	1987	1988	1989
y	745	859	980	1114	1259

x	1990	1991	1992	1993	1994
y	1399	1568	1714	1850	1953

In Exercises 27 and 28, make a table of values for $x = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$, and 2. Then plot the points on a rectangular coordinate system.

27. $y = 2 - \frac{1}{2}x$ **28.** $y = 2 - \frac{1}{2}x^2$

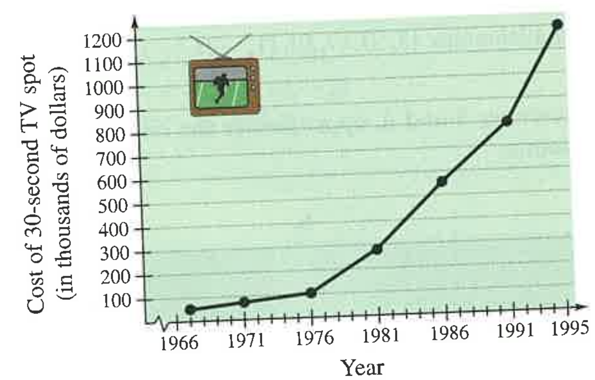
Milk Prices **In Exercises 29 and 30, refer to the figure, which shows the average price in dollars paid to farmers for milk.** (Source: U.S. Department of Agriculture and the National Milk Producers Federation)



29. Approximate the highest price of milk shown in the graph. When did this occur?

30. Approximate the percent drop in the price of milk from the highest price shown in the graph to the price paid to farmers in January, 1993.

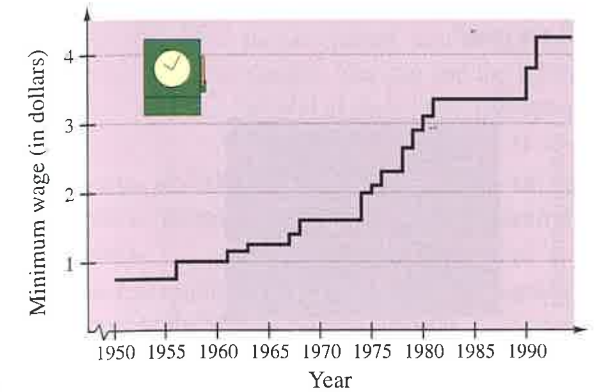
TV Advertising **In Exercises 31 and 32, refer to the figure.** (Source: Nielson Media Research)



31. Approximate the percent increase in the cost of a 30-second spot from Super Bowl I in 1967 to Super Bowl XXIX in 1995.

32. Estimate the increase in cost of a 30-second spot (a) from Super Bowl V to Super Bowl XV, and (b) from Super Bowl XV to Super Bowl XXV.

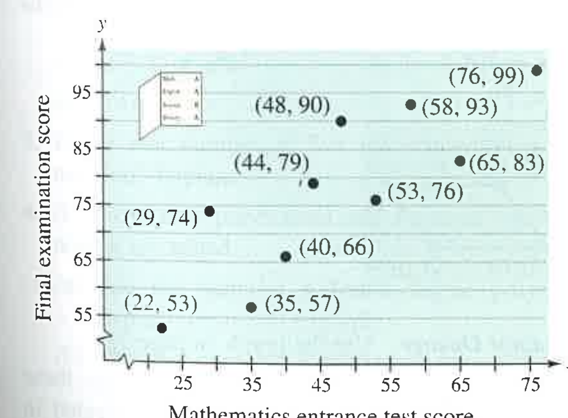
Minimum Wage **In Exercises 33 and 34, refer to the figure.** (Source: U.S. Department of Labor)



33. During which decade did the minimum wage increase most rapidly?

34. Approximate the percent increase in the minimum wage from 1990 to 1994.

Data Analysis **In Exercises 35 and 36, refer to the figure, which shows the mathematics entrance test scores x, and the final examination scores y, in an algebra course for a sample of 10 students.**



35. Find the entrance exam score of any student with a final exam score in the 80's.

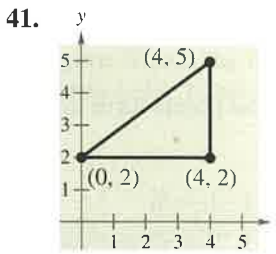
36. Does a higher entrance exam score imply a higher final exam score? Explain.

In Exercises 37–40, find the distance between the points. (*Note*: In each case the two points lie on the same horizontal or vertical line.)

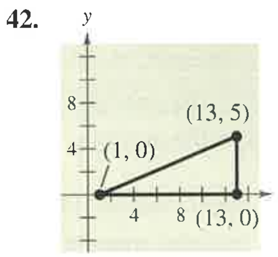
37. $(6, -3), (6, 5)$
38. $(1, 4), (8, 4)$
39. $(-3, -1), (2, -1)$
40. $(-3, -4), (-3, 6)$

In Exercises 41–44, (a) find the length of each side of a right triangle, and (b) show that these lengths satisfy the Pythagorean Theorem.

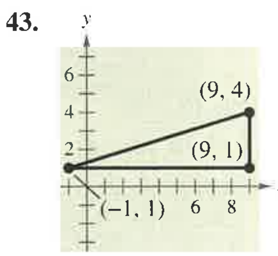
41.



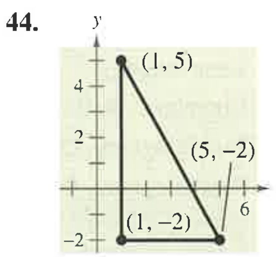
42.



43.



44.



In Exercises 45–56, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

45. $(1, 1), (9, 7)$
46. $(1, 12), (6, 0)$
47. $(-4, 10), (4, -5)$
48. $(-7, -4), (2, 8)$
49. $(-1, 2), (5, 4)$
50. $(2, 10), (10, 2)$
51. $(\frac{1}{2}, 1), (-\frac{5}{2}, \frac{4}{3})$
52. $(-\frac{1}{3}, -\frac{1}{3}), (-\frac{1}{6}, -\frac{1}{2})$
53. $(6.2, 5.4), (-3.7, 1.8)$
54. $(-16.8, 12.3), (5.6, 4.9)$
55. $(-36, -18), (48, -72)$
56. $(1.451, 3.051), (5.906, 11.360)$

In Exercises 57 and 58, use the Midpoint Formula to estimate the sales of a company in 1993, given the sales in 1991 and 1995. Assume the sales followed a linear pattern.

57.

Year	1991	1995
Sales	\$520,000	\$740,000

58.

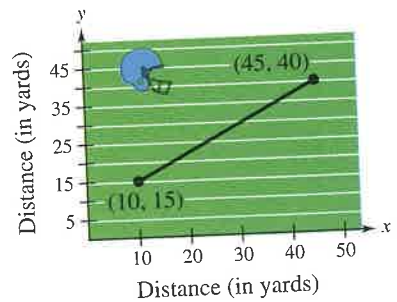
Year	1991	1995
Sales	\$4,200,000	\$5,650,000

In Exercises 59–64, show that the points form the vertices of the polygon.

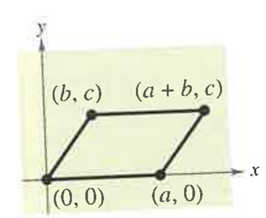
59. Right triangle: $(4, 0), (2, 1), (-1, -5)$
60. Isosceles triangle: $(1, -3), (3, 2), (-2, 4)$
61. Rhombus: $(0, 0), (1, 2), (2, 1), (3, 3)$ (A rhombus is a parallelogram whose sides are all the same length.)
62. Rhombus: $(4, 0), (0, 6), (-4, 0), (0, -6)$
63. Parallelogram: $(2, 5), (0, 9), (-2, 0), (0, -4)$
64. Parallelogram: $(0, 1), (3, 7), (4, 4), (1, -2)$

65. A line segment has (x_1, y_1) as one endpoint and (x_m, y_m) as its midpoint. Find the other endpoint (x_2, y_2) of the line segment in terms of x_1, y_1, x_m, and y_m.
66. Use the result of Exercise 65 to find the coordinates of the endpoint of a line segment if the coordinates of the other endpoint and midpoint are, respectively,
(a) $(1, -2), (4, -1)$. (b) $(-5, 11), (2, 4)$.
67. Use the Midpoint Formula three times to find the three points that divide the line segment joining (x_1, y_1) and (x_2, y_2) into four parts.
68. Use the result of Exercise 67 to find the points that divide the line segment joining the given points into four equal parts.
(a) $(1, -2), (4, -1)$ (b) $(-2, -3), (0, 0)$

69. ***Football Pass*** In a football game, a quarterback throws a pass from the 15-yard line, 10 yards from the sideline (see figure). The pass is caught on the 40-yard line, 45 yards from the same sideline. How long is the pass?



70. ***Flying Distance*** A plane flies in a straight line to a city that is 100 kilometers east and 150 kilometers north of the point of departure. How far does it fly?
71. ***Make a Conjecture*** Plot the points $(2, 1), (-3, 5)$, and $(7, -3)$ on a rectangular coordinate system. Then change the sign of the x-coordinate of each point and plot the three new points on the same rectangular coordinate system. What conjecture can you make about the location of a point when the sign of the x-coordinate is changed?
72. Prove that the diagonals of the parallelogram in the figure bisect each other.

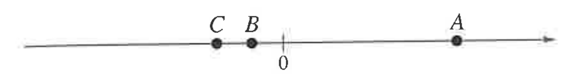


73. ***Chapter Opener*** Use the graph on page 1.
(a) Describe any trends in the data. From these trends, predict the number of artists elected in 1996.
(b) Why do you think the numbers elected in 1986 and 1987 were greater than in other years?

FOCUS ON CONCEPTS

In this chapter, you studied several concepts that are required in the study of algebra. You can use the following questions to check your understanding of several of these basic concepts. The answers to these questions are given in the back of the book.

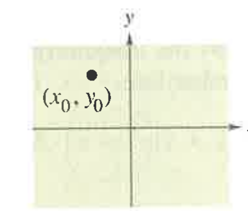
1. Describe the differences among the sets of natural numbers, integers, rational numbers, and irrational numbers.
2. Three real numbers are shown on the real number line. Determine the sign of each expression.

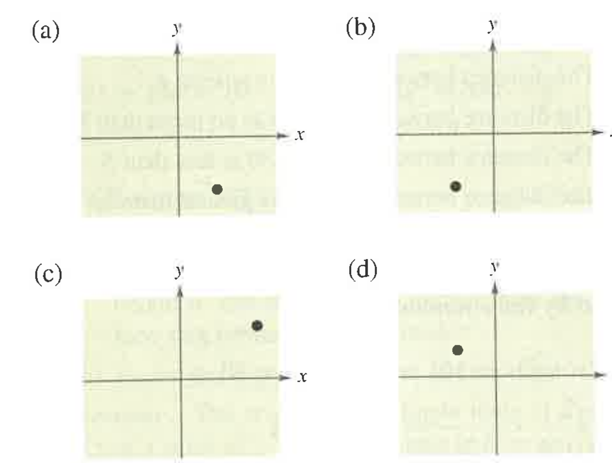


(a) $-A$ (b) $-C$
(c) $B - A$ (d) $A - C$

3. You may hear it said that to take the absolute value of a real number you simply remove any negative sign and make the number positive. Can it ever be true that $|a| = -a$ for a real number a? Explain.
4. Explain why each of the following is *not* equal.
(a) $(3x)^{-1} \neq \dfrac{3}{x}$ (b) $y^3 \cdot y^2 \neq y^6$
(c) $(a^2b^3)^4 \neq a^6b^7$ (d) $(a + b)^2 \neq a^2 + b^2$
(e) $\sqrt{4x^2} \neq 2x$ (f) $\sqrt{2} + \sqrt{3} \neq \sqrt{5}$
5. Is the real number 52.7×10^5 written in scientific notation? Explain.
6. A third-degree polynomial and a fourth-degree polynomial are added.
(a) Can the sum be a fourth-degree polynomial? Explain or give an example.
(b) Can the sum be a second-degree polynomial? Explain or give an example.
(c) Can the sum be a seventh-degree polynomial? Explain or give an example.
7. Explain what is meant when it is said that a polynomial is in factored form.
8. How do you determine whether a rational expression is in reduced form?

In Exercises 9–12, use the plot of the point (x_0, y_0) in the figure. Match the transformation of the point with the correct plot. [The plots are labeled (a), (b), (c), and (d).]





9. $(x_0, -y_0)$ 10. $(-2x_0, y_0)$
11. $(x_0, \frac{1}{2}y_0)$ 12. $(-x_0, -y_0)$

Review Exercises

In Exercises 1 and 2, determine which numbers in the set are (a) natural numbers, (b) integers, (c) rational numbers, and (d) irrational numbers.

1. $\{11, -14, -\frac{8}{9}, \frac{5}{2}, \sqrt{6}, 0.4\}$
2. $\{\sqrt{15}, -22, -\frac{10}{3}, 0, 5.2, \frac{3}{7}\}$

In Exercises 3 and 4, use a calculator to find the decimal form of the rational number. If it is a nonterminating decimal, write the repeating pattern.

3. (a) $\frac{5}{6}$ (b) $\frac{7}{8}$
4. (a) $\frac{9}{25}$ (b) $\frac{5}{7}$

In Exercises 5 and 6, give a verbal description of the subset of real numbers represented by the inequality, and sketch the subset on the real number line.

5. $x \leq 7$
6. $x > 1$

In Exercises 7–10, use absolute value notation to describe the expression.

7. The distance between x and 7 is at least 4.
8. The distance between x and 25 is no more than 10.
9. The distance between y and -30 is less than 5.
10. The distance between y and $\frac{1}{2}$ is greater than 2.

In Exercises 11–14, identify the rule of algebra illustrated by the equation.

11. $2x + (3x - 10) = (2x + 3x) - 10$
12. $\dfrac{2}{y+4} \cdot \dfrac{y+4}{2} = 1, \quad y \neq -4$
13. $0 + (a - 5) = a - 5$
14. $(t + 4)(2t) = (2t)(t + 4)$

In Exercises 15–18, simplify the expression.

15. (a) $(-2z)^3$ (b) $(a^2b^4)(3ab^{-2})$
16. (a) $\dfrac{(8y)^0}{y^2}$ (b) $\dfrac{40(b-3)^5}{75(b-3)^2}$
17. (a) $\dfrac{6^2u^3v^{-3}}{12u^{-2}v}$ (b) $\dfrac{3^{-4}m^{-1}n^{-3}}{9^{-2}mn^{-3}}$
18. (a) $(x + y^{-1})^{-1}$ (b) $\left(\dfrac{x^{-3}}{y}\right)\left(\dfrac{x}{y}\right)^{-1}$

In Exercises 19 and 20, write the number in scientific notation.

19. *1994 Net Sales of Procter and Gamble Company:* \$30,296,000,000 (Source: 1994 Annual Report)
20. *Number of Meters in One Foot:* 0.3048

In Exercises 21 and 22, write the number in decimal form.

21. *Distance Between Sun and Jupiter:* 4.833×10^8 miles
22. *Ratio of Day to Year:* 2.74×10^{-3}

In Exercises 23 and 24, use a calculator to evaluate the expression. (Round your answer to three decimal places.)

23. (a) $1800(1 + 0.08)^{24}$
(b) $0.0024(7,658,400)$
24. (a) $50,000\left(1 + \dfrac{0.075}{12}\right)^{48}$
(b) $\dfrac{28,000,000 + 34,000,000}{87,000,000}$

In Exercises 25 and 26, fill in the missing expression.

	Radical Form	*Rational Exponent Form*
25.	$\sqrt{16} = 4$	$\quad = 4$
26.	$\quad = 2$	$16^{1/4} = 2$

In Exercises 27 and 28, simplify by removing all possible factors from the radical.

27. (a) $\sqrt{4x^4}$ (b) $\sqrt{\dfrac{18u^2}{b^3}}$
28. (a) $\sqrt[3]{\dfrac{2x^3}{27}}$ (b) $\sqrt[5]{64x^6}$

In Exercises 29 and 30, rewrite the expression by rationalizing the denominator. Simplify your answer.

29. $\dfrac{1}{2 - \sqrt{3}}$
30. $\dfrac{1}{\sqrt{x} - 1}$

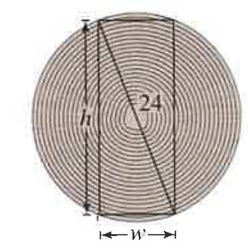
In Exercises 31 and 32, simplify the expression.

31. $\sqrt{50} - \sqrt{18}$
32. $\sqrt{8x^3} + \sqrt{2x}$

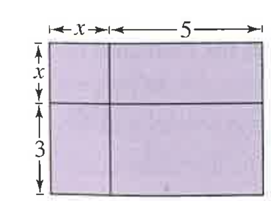
33. ***Strength of a Wooden Beam*** The rectangular cross section of a wooden beam cut from a log of diameter 24 inches (see figure) will have a maximum strength if its width w and height h are given by

$w = 8\sqrt{3}$ and $h = \sqrt{24^2 - (8\sqrt{3})^2}$.

Find the area of the rectangular cross section and express the answer in simplest form.



34. ***Geometric Modeling*** Use the area model to write two expressions for the total area in the figure. Then equate the two expressions and name the algebraic property illustrated.



In Exercises 35–46, describe and correct the error.

35. $\cancel{10(4 \cdot 7) = 40 \cdot 70}$
36. $\cancel{\left(\frac{1}{3}x\right)\left(\frac{1}{3}y\right) = \frac{1}{3}xy}$
37. $\cancel{4\left(\frac{3}{7}\right) = \frac{12}{28}}$
38. $\cancel{\frac{2}{9} \times \frac{4}{9} = \frac{8}{9}}$
39. $\cancel{\dfrac{x-1}{1-x} = 1}$
40. $\cancel{(2x)^4 = 2x^4}$
41. $\cancel{(-x)^6 = -x^6}$
42. $\cancel{(3^4)^4 = 3^8}$
43. $\cancel{\sqrt{3^2 + 4^2} = 3 + 4}$
44. $\cancel{(5 + 8)^2 = 5^2 + 8^2}$
45. $\cancel{\sqrt{10x} = 10\sqrt{x}}$
46. $\cancel{\sqrt{7x}\sqrt[3]{2} = \sqrt{14x}}$

In Exercises 47–52, perform the operations and write the result in standard form.

47. $-(3x^2 + 2x) + (1 - 5x)$
48. $8y - [2y^2 - (3y - 8)]$
49. $(2x - 3)^2$
50. $(3\sqrt{5} + 2)(3\sqrt{5} - 2)$
51. $(x^3 - 3x)(2x^2 + 3x + 5)$
52. $\left(x - \dfrac{1}{x}\right)(x + 2)$

In Exercises 53–58, factor completely.

53. $x^3 - x$
54. $x(x - 3) + 4(x - 3)$
55. $2x^2 + 21x + 10$
56. $3x^2 + 14x + 8$
57. $x^3 - x^2 + 2x - 2$
58. $x^3 - 1$

59. ***Exploration*** The surface area of a right circular cylinder is $S = 2\pi r^2 + 2\pi rh$.
(a) Draw a right circular cylinder of radius r and height h. Use the figure to explain how the surface area formula was obtained.
(b) Factor the expression for the surface area.

60. ***Revenue*** The revenue for selling x units of a product at a price of p dollars per unit is $R = xp$. For a particular product the revenue is

$R = 1600x - 0.50x^2$.

Factor the expression, and determine an expression that gives the price in terms of x.

In Exercises 61 and 62, insert the missing factor.

61. $\frac{2}{3}x^4 - \frac{3}{8}x^3 + \frac{5}{6}x^2 = \frac{1}{24}x^2(\quad)$

62. $\frac{t}{\sqrt{t+1}} - \sqrt{t+1} = \frac{1}{\sqrt{t+1}}(\quad)$

In Exercises 63–68, perform the operations and simplify.

63. $\frac{x^2-4}{x^4-2x^2-8} \cdot \frac{x^2+2}{x^2}$

64. $\frac{4x-6}{(x-1)^2} \div \frac{2x^2-3x}{x^2+2x-3}$

65. $2x + \frac{3}{2(x-4)} - \frac{1}{2(x+2)}$

66. $\frac{1}{x} - \frac{x-1}{x^2+1}$

67. $\frac{1}{x-1} + \frac{1-x}{x^2+x+1}$

68. $\frac{1}{L}\left(\frac{1}{y} - \frac{1}{L-y}\right)$, where L is a constant

In Exercises 69 and 70, simplify the compound fraction.

69. $\dfrac{\left[\dfrac{3a}{(a^2/x)-1}\right]}{\left(\dfrac{a}{x}-1\right)}$

70. $\dfrac{\left(\dfrac{1}{2x-3} - \dfrac{1}{2x+3}\right)}{\left(\dfrac{1}{2x} - \dfrac{1}{2x+3}\right)}$

In Exercises 71-100, solve the equation (if possible) and check your solution.

71. $3x - 2(x+5) = 10$

72. $4x + 2(7-x) = 5$

73. $4(x+3) - 3 = 2(4-3x) - 4$

74. $\frac{1}{2}(x-3) - 2(x+1) = 5$

75. $3\left(1 - \frac{1}{5t}\right) = 0$

76. $\frac{1}{x-2} = 3$

77. $6x = 3x^2$

78. $15 + x - 2x^2 = 0$

79. $(x+4)^2 = 18$

80. $16x^2 = 25$

81. $x^2 - 12x + 30 = 0$

82. $x^2 + 6x - 3 = 0$

83. $5x^4 - 12x^3 = 0$

84. $4x^3 - 6x^2 = 0$

85. $\frac{4}{(x-4)^2} = 1$

86. $\frac{1}{(t+1)^2} = 1$

87. $\sqrt{x+4} = 3$

88. $\sqrt{x-2} - 8 = 0$

89. $2\sqrt{x} - 5 = 0$

90. $\sqrt{3x-2} = 4 - x$

91. $\sqrt{2x+3} + \sqrt{x-2} = 2$

92. $5\sqrt{x} - \sqrt{x-1} = 6$

93. $(x-1)^{2/3} - 25 = 0$

94. $(x+2)^{3/4} = 27$

95. $(x+4)^{1/2} + 5x(x+4)^{3/2} = 0$

96. $8x^2(x^2-4)^{1/3} + (x^2-4)^{4/3} = 0$

97. $|x-5| = 10$

98. $|2x+3| = 7$

99. $|x^2-3| = 2x$

100. $|x^2-6| = x$

In Exercises 101–104, solve the equation for the indicated variable.

101. Solve for r: $V = \frac{1}{3}\pi r^2 h$

102. Solve for X: $Z = \sqrt{R^2 - X^2}$

103. Solve for p: $L = \frac{k}{3\pi r^2 p}$

104. Solve for v: $E = 2kw\left(\frac{v}{2}\right)^2$

In Exercises 105–116, solve the inequality.

105. $x^2 - 2x \geq 3$

106. $\frac{1}{2}(3-x) > \frac{1}{3}(2-3x)$

107. $\frac{x-5}{3-x} < 0$

108. $\frac{2}{x+1} \leq \frac{3}{x-1}$

109. $|x-2| < 1$

110. $|x| \leq 4$

111. $|x - \frac{3}{2}| \geq \frac{3}{2}$

112. $|x-3| > 4$

113. $\frac{x}{5} - 6 \leq -\frac{x}{2} + 6$

114. $2x^2 + x \geq 15$

115. $(x-4)|x| > 0$

116. $|x(x-6)| < 5$

In Exercises 117 and 118, find the domain of the expression by finding the interval(s) on the real number line for which the radicand is nonnegative.

117. $\sqrt{2x-10}$

118. $\sqrt{x(x-4)}$

Geometry **In Exercises 119 and 120, plot the points and verify that the points form the polygon.**

119. *Right Triangle:* $(2, 3), (13, 11), (5, 22)$

120. *Parallelogram:* $(1, 2), (8, 3), (9, 6), (2, 5)$

In Exercises 121–124, determine the quadrant(s) in which (x, y) is located so that the conditions are satisfied.

121. $x > 0$ and $y = -2$

122. $y > 0$

123. $(-x, y)$ is in the third quadrant.

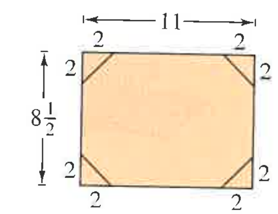
124. $xy = 4$

In Exercises 125 and 126, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

125. $(-3, 8), (1, 5)$

126. $(5.6, 0), (0, 8.2)$

127. ***Geometry*** The four corners are cut from an $8\frac{1}{2}$-by-11-inch piece of paper (see figure). Find the perimeter of the remaining piece of paper.



128. Complete the table.

n	1	10	10^2	10^4	10^6	10^{10}
$\frac{5}{\sqrt{n}}$						

What number is $5/\sqrt{n}$ approaching as n increases without bound?

129. Let m and n be any two integers. Then $2m$ and $2n$ are even integers and $(2m+1)$ and $(2n+1)$ are odd integers.

(a) Prove that the sum of two even integers is even.

(b) Prove that the sum of two odd integers is even.

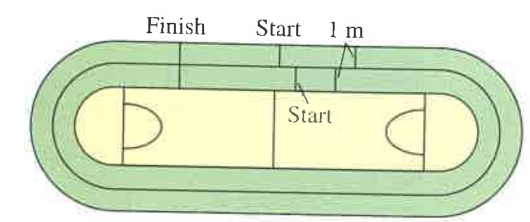
(c) Prove that the product of an even integer and any integer is even.

130. ***Monthly Profit*** In October, a company's total profit was 12% more than it was in September. The total profit for the two months was \$689,000. Find the profit for each month.

131. ***Discount Rate*** The price of a television set has been discounted \$85. The sale price is \$340. What was the percent discount?

132. ***Mixture Problem*** A car radiator contains 10 liters of a 30% antifreeze solution. How many liters will have to be replaced with pure antifreeze if the resulting solution is to be 50% antifreeze?

133. ***Running Track*** A fitness center has two running tracks around a rectangular playing floor (see figure). The tracks are 1 meter wide and form semicircles at the narrow ends of the floor. How much longer is the running distance on the outer track than on the inner track?



134. ***Cost Sharing*** A group agrees to share equally in the cost of a \$48,000 piece of machinery. If they can find two more group members, each member's share will decrease by \$4000. How many are presently in the group?

135. ***Venture Capital*** You are planning to start a small business that will require an investment of \$90,000. You have found some people who are willing to share equally in the venture. If you can find three more people, each person's share will decrease by \$2500. How many people have you found so far?

136. ***Average Speed*** You commute 56 miles one way to work. The trip to work takes 10 minutes longer than the trip home. Your average speed on the trip home is 8 miles per hour faster. What is your average speed on the trip home?